International Journal of Innovation and Applied Studies
ISSN 2028-9324 Vol. 18 No. 1 Oct. 2016, pp. 262-267

http://www.ijias.issr-journals.org/

# Sumudu Transform Series Decomposition Method for Solving Nonlinear Volterra Integro-Differential Equations

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**ABSTRACT:** In this paper, Sumudu Transform Series Decomposition Method (STSDM) for solving Integro-Differential Equation is presented. The method is an elegant combination of Sumudu Transform method, series expansion and Adomian polynomial. Three numerical problems were solved and compared with the exact solutions and the results by other approximate methods in order to check the effectiveness, reliability, accuracy, and the convergence of the method. The results obtained by STSDM showed that it is a powerful mathematical technique for solving wide range of physical problems arising in science and engineering fields.

**KEYWORDS:** Adomianl Decomposition, Volterra Integro-Differential equation, Series expansion, Sumudu Transform.

## 1 INTRODUCTION

The Volterra integro-differential equation arises from the mathematical modeling of the spatio-temporal development of an epidemics model in addition to various physical and biological models [1], [2].

The general nonlinear Volterra integro-differential equation is given as

$$u^{(n)}(x) = f(x) + \int_0^x k(x,t)\left[R(u(t)) + N(u(t))\right] \qquad (1)$$

$$u^{(k)}(0) = b_k, \qquad 0 \le k \le (n-1)$$

Where $u^{(n)}(x)$ is the nth derivative of the unknown function u(x) that is to be determined, k(x,t) is the kernel of the integral equation, f(x) is a known analytic function, R(u) and N(u) are linear and nonlinear functions, respectively. For n=0, (1) turns out to be a classical nonlinear integral equation.

Sumudu Transform is an integral-based transform named by Watugala [3]. Since the formulation of Sumudu Transform, many works based on this transform were being done [4], [6].

Sumudu Transform is written as

$$F(u) = S\left[f(x)\right] = \int_0^\infty \frac{1}{u} f(x) e^{-x/u} dx, \qquad (2)$$

for any f(x) and the Sumudu Transform of $n^{th}$ derivative is also given as

$$S\left[\frac{d^n f(t)}{dt^n}\right] = \frac{1}{u^n}\left[F(u) - \sum_{k=0}^{n-1} u^k \left.\frac{d^n f(t)}{dt^n}\right|_{t=0}\right] \quad (3)$$

Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point

In the 1980s George Adomian introduced a new method to solve nonlinear differential equations [7], [9]. This method has since been termed the Adomian Decomposition Method (ADM) and has been the subject of many investigations [10], [11], [12], [13]. The ADM involves separating the equation under investigation into linear and nonlinear portions. The linear operator representing the linear portion of the equation is inverted and the inverse operator is then applied to the equation under any considerable given conditions. The nonlinear portion is decomposed into a series of special polynomials called Adomian polynomials.

In recent years, many new methods, such as He's Homotopy Perturbation Method [2], Modified Homotopy Perturbation Method [14], Cubic Spline Collocation Method [15], Adomian Decomposition and Variational Iteration Methods [16] and many more are used to solve integra and integro-differential equation problems. Here in this paper, a new approximate method called Sumudu Transform Series Decomposition Method (STSDM) is proposed to solve both second and fourth order nonlinear Volterra integro-differential equations and the comparisons of results obtained by STSDM are made with the exact and modified homotopy perturbation method.

## 2 THE METHOD

### 2.1 DERIVATION OF THE SUMUDU TRANSFORM SERIES DECOMPOSITION METHOD (STSDM)

Given a general nonlinear non-homogeneous differential equation

$$LU(x) + RU(x) + NU(x) = g(x) \quad (4)$$

Where L is the highest order linear differential operator, R is the linear differential operator of order less than L, N is the nonlinear differential operator, U is the dependent variable, x is an independent variable and g(x) is the source term which is assumed to have series expansion.

Application of the Sumudu Transform on equation (4) resulted into

$$S[LU(x)] + S[RU(x)] + S[NU(x)] = S[g(x)] \quad (5)$$

Using the differentiation property of the Sumudu transform (3) in (5) to have

$$\frac{S[U(x)]}{u^m} - \sum_{k=o}^{m-1} \frac{U(x)^{(k)}(0)}{u^{(m-k)}} + S[RU(x)] + S[NU(x)] = S[g(x)] \quad (6)$$

Where

$$\sum_{k=o}^{m-1} \frac{U(x)^{(k)}(0)}{u^{(m-k)}} = \sum_{k=o}^{m-1} \frac{U(0)^{(k)}}{u^{(m-k)}}$$

Further simplification of (6) gave

$$S[U(x)] - u^m \sum_{k=o}^{m-1} \frac{U(x)^{(k)}(0)}{u^{(m-k)}} + u^m [S[RU(x)] + S[NU(x)] - S[g(x)]] = 0 \quad (7)$$

Where S denotes the sumudu transform,

Application of Sumudu inverse Transform on (7) yielded

$$U(x) = G(x) - S^{-1}\left[u^m \left[[S[RU(x)] + S[NU(x)]]\right]\right] \quad (8)$$

And,

$$G(x) = S^{-1}\left[u^m\left[\sum_{k=o}^{m-1}\frac{U(x)^{(k)}(0)}{u^{(m-k)}} + S[g(x)]\right]\right] \tag{9}$$

Where G(x) represents the term arising from the source term and the prescribed initial conditions.

The representation of the solution (8) as an infinite series is given below:

$$U(x) = \sum_{n=o}^{\infty} U_n(x) \tag{10}$$

The nonlinear term is being decomposed as:

$$NU(x) = \sum_{r=0}^{\infty} A_n(u_0, u_1, \ldots u_n) \tag{11}$$

Where $A_n$ are the Adomian polynomials of functions $U_0$, $U_1$, $U_2 \ldots U_n$ and can be calculated by formula given in [17] as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{r=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \quad n = 0,1,2,\ldots \tag{12}$$

Substituting (10) and (11) into (8) yielded

$$\sum_{n=o}^{\infty} U_{n+1}(x) = G(x) - S^{-1}\left[Su^m\left[[[R\sum_{n=o}^{\infty}U_n(x)] + [\sum_{n=o}^{\infty}A_n]]\right]\right] \tag{13}$$

Simplification of equation (13) as many times as possible resulted into series solution

$$\left.\begin{aligned} U_0(x) &= G(x) = S^{-1}\left[u^m\left[\sum_{k=o}^{m-1}\frac{U(x)^{(k)}(0)}{u^{(m-k)}} + S[g(x)]\right]\right] \\ U_{n+1}(x) &= -S^{-1}\left[Su^m\left[[[RU_n(x)] + [A_n]]\right]\right] \end{aligned}\right\} \quad n \geq 0 \tag{15}$$

Equation (14) led to the generally recursive relation given by:

$$\left.\begin{aligned} U_0(x) &= G(x) = S^{-1}\left[u^m\left[\sum_{k=o}^{m-1}\frac{U(x)^{(k)}(0)}{u^{(m-k)}} + S[g(x)]\right]\right] \\ U_{n+1}(x) &= -S^{-1}\left[Su^m\left[[[RU_n(x)] + [A_n]]\right]\right] \end{aligned}\right\} \quad n \geq 0 \tag{15}$$

when the Sumudu Transform and the Sumudu inverse Transform are applied on (15) respectively, the iteration $U_0$, $U_1$, $U_2 \ldots U_n$ were obtained, which in turn gave the general solution as

$$U(x) = U_0(x) + U_1(x) + U_2(x) + U_3(x) + \ldots \tag{16}$$

## 3 NUMERICAL APPLICATION

Example 1

Consider the second order nonlinear Volterra integro-differential equation

$$u''(x) = \sinh x + \frac{1}{2}\cosh x \sinh x - \frac{1}{2}x - \int_0^x u^2(t)dt \tag{17}$$

Subject to the initial condition

$$u(0) = 0, \qquad u'(0) = 1 \tag{18}$$

Using STSDM to solve (17) alongside the initial condition (18) to obtain the approximate solution

$$u(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \ldots \tag{19}$$

Equation (19) can also be written in a closed form as

$$u(x) = \sinh x \tag{20}$$

Equation (20) above is the same as the exact solution of (17)

Example 2

Consider the fourth order nonlinear initial value problem of Volterra integro-differential equation

$$\left.\begin{aligned} &U^{(4)}(x) = f(x) + 3\int_0^x U^3(t)dt, \\ &U(0) = U''(0) = 1, U'(0) = U'''(0) = -1 \\ &f(x) = e^{-x} + e^{-3x} - 1 \end{aligned}\right\} \tag{21}$$

***Table 1: Comparison of the results obtained by STSDM with the Exact and Modified Homotopy Perturbation Method for example 2***

| X | Exact | MHPM | STSDM | MHPM error | STSDM error |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0.00000 | 0.00000 |
| 0.04 | 0.9607894392 | 0.9608106692 | 1.105170969 | $2.12300*10^{-05}$ | $4.76800*10^{-11}$ |
| 0.08 | 0.9231163464 | 0.9232854120 | 1.221403172 | $1.690656*10^{-04}$ | $1.33606*10^{-11}$ |
| 0.12 | 0.8869204367 | 0.8874885866 | 1.349860199 | $5.681499*10^{-04}$ | $1.71535*10^{-11}$ |
| 0.16 | 0.8521437890 | 0.8534850921 | 1.491827980 | $1.341303*10^{-03}$ | $3.388454*10^{-11}$ |
| 0.2 | 0.8187307531 | 0.8213405980 | 1.648727592 | $2.609845*10^{-03}$ | $2.226369*10^{-11}$ |
| 0.24 | 0.7866278611 | 0.7911217470 | 1.822129337 | $4.493886*10^{-03}$ | $3.686180*10^{-11}$ |
| 0.28 | 0.7557837415 | 0.7628963355 | 2.013768118 | $7.112594*10^{-03}$ | $5.969278*10^{-11}$ |
| 0.32 | 0.7261490371 | 0.7367334755 | 2.225560096 | $1.058444*10^{-02}$ | $8.040831*10^{-11}$ |
| 0.36 | 0.6976763261 | 0.7127037390 | 2.459620431 | $1.150274*10^{-02}$ | $1.964534*10^{-10}$ |

Example 2

Consider the fourth order nonlinear boundary value problem of Volterra integro-differential equation

$$\left.\begin{aligned} &U^{(4)}(x) = f(x) - \int_0^x U(t)U''(t)dt, \\ &U(0) = U'(0) = U''(0) = 1,\ U(1) = e \\ &f(x) = e^x - \frac{1}{2}e^{2x} + \frac{1}{2} \end{aligned}\right\} \tag{22}$$

Since (22) is an initial boundary value problem (IBVP), an assumption is made that

$$u'''(0) = k \tag{23}$$

Having obtained the general solution $u(x)$ using STSDM, the boundary condition $u(1)=e$ is imposed on it in order to find the value $k=1.000309600$

***Table 2: Comparison of the results obtained by STSDM with the Exact and Modified Homotopy Perturbation Method for example 3***

| X | Exact | MHPM | STSDM | MHPM error | STSDM error |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0.00000 | 0.00000 |
| 0.1 | 1.105170918 | 1.105230748 | 1.105170969 | $5.9830*10^{-05}$ | $5.1*10^{-08}$ |
| 0.2 | 1.221402758 | 1.221843785 | 1.221403172 | $4.41027*10^{-04}$ | $4.14*10^{-07}$ |
| 0.3 | 1.349858808 | 1.351209687 | 1.349860199 | $1.350879*10^{-03}$ | $1.391*10^{-06}$ |
| 0.4 | 1.491824698 | 1.494673169 | 1.491827980 | $2.848471*10^{-03}$ | $3.282*10^{-06}$ |
| 0.5 | 1.648721271 | 1.653535684 | 1.648727592 | $4.814413*10^{-03}$ | $6.321*10^{-06}$ |
| 0.6 | 1.822118800 | 1.829034189 | 1.822129337 | $6.915389*10^{-03}$ | $1.054*10^{-05}$ |
| 0.7 | 2.013752707 | 2.022315475 | 2.013768118 | $8.562768*10^{-03}$ | $1.541*10^{-05}$ |
| 0.8 | 2.225540928 | 2.234405354 | 2.225560096 | $8.864426*10^{-03}$ | $1.917*10^{-05}$ |
| 0.9 | 2.459603111 | 2.466171899 | 2.459620431 | $6.568788*10^{-03}$ | $1.732*10^{-05}$ |
| 1.0 | 2.718281828 | 2.718281826 | 2.718281830 | $2.0*10^{-09}$ | $2.0*10^{-09}$ |

## 4 CONCLUSION

This paper has shown that the approximate solution of nonlinear Volterra integro-differential equations can be accurately obtained using Sumudu Transform Series Decomposition Method (STSDM). The comparison of the results obtained using STSDM with the exact and modified perturbation method (MHPM) is a clear indication that the proposed method is a powerful and easy to use approximate method for solving nonlinear Volterra integro-differential equations, it also attests to the fact that STSDM is a better, accurate, efficient, effective and reliable method for solving nonlinear integral systems and equations used in sciences and engineering discipline.

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