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A Boundedness Theorem for a Certain Third Order Nonlinear Differential Equation

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Abstract: In this study, we establish conditions for ultimate boundedness of solutions for a certain third order nonlinear differential equation using a complete Yoshizawa function. The result includes and extends the earlier results in the literature.

Keywords: Boundedness of solutions, nonlinear third order equation, complete Yoshizawa functions, incomplete Lyapunov functions

INTRODUCTION

It is well known from relevant literature that there have been deep and thorough studies on the qualitative behaviour of solutions of third order nonlinear differential equations in recent years. Mean while, many articles have been devoted to the investigation of boundedness of solutions for various forms of third order nonlinear differential equations; see for instance Yoshizawa [20] and Reissiq *et. al.*,[14] which are background and survey books respectively, and Ademola *et. al.*,[1],Afuwape [2], Andres [3], Bereketoğlu and Györi [4], Chukwu [5], Ezeilo [6, 7, 8, 9, 10, 11], Ezeilo and Tejumola [12], Hara [14], Swick [16, 17], Tejumola [18], and Tunç [19]. These works were done with the aid of complete Yoshizawa and incomplete Lyapunov functions.

Our observation in relevant literature reveals that works on the ultimate boundedness of solutions for the third order nonlinear differential equation (1) using a complete Yoshizawa function are scarce. The purpose of this paper is to study the ultimate boundedness of solutions of the third order nonlinear non-autonomous ordinary differential equation

$$\dddot{x} + f(x,\dot{x},\ddot{x})\ddot{x} + g(x,\dot{x}) + h(x,\dot{x},\ddot{x}) = p(t,x,\dot{x},\ddot{x}) \qquad (1)$$

On setting $\dot{x} = y$ and $\ddot{x} = z$ equation (1) is equivalent to the system of differential equations

$$\dot{x} = y, \qquad \dot{y} = z,$$
$$\dot{z} = p(t,x,y,z) - f(x,y,z)z - g(x,y) - h(x,y,z) \qquad (2)$$

In which $f, h \in C(\mathbb{R}^3, \mathbb{R})$, $p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $\mathbb{R} = [0,\infty)$ and $\mathbb{R} = [-\infty,\infty)$.

It is supposed that the functions f, g, h and p depend only on the arguments displayed explicitly, and the dots, as usual, dente differentiation with respect to the independent variable t. The partial derivatives

$\frac{\partial f(x,y,z)}{\partial x} = f_x(x,y,z)$, $\frac{\partial g(x,y)}{\partial x} = g_x(x,y)$, $\frac{\partial h(x,y,z)}{\partial x} = h_x(x,y,z)$, $\frac{\partial h(x,y,z)}{\partial y} = h_y(x,y,z)$ and $\frac{\partial h(x,y,z)}{\partial z} = h_z(x,y,z)$ exist and are continuous. Here we shall use a complete Yoshizawa function as our basic tool to achieve the desired result.

RESULT AND DISCUSSION

Theorem: In addition to the basic assumptions on f, g, h and p, suppose there are positive constants $a, b, c, \delta_0, \delta_1, \delta_2, \Delta_0$ and M such that the following conditions are satisfied

- $a \le f(x,y,z) \le \delta_0 |h(x,y,z)|$ for all x, y;
- $0 \le g(x,y)\,\mathrm{sgn}\,y$, for all x and y, $b \le \frac{g(x,y)}{y}$, $(|y| \ge \Delta_0)$ for all x and $\max_{|y| \le \Delta_0} |g(x,y)| \le \delta_1 |h(x,0,0)|$;
- $H_x(x,0,0) \le c$ for all x, and $0 < c < ab$;
- $h(x,y,z)\,\mathrm{sgn}\,x \to +\infty$ as $|x| \to \infty$;
- $g_x(x,y) \le 0, y f_x(x,y,0) \le 0$ for all x, y;
- $(x,y,0) \ge 0, h_z(x,y,z) \ge 0$, $yzh_z(x,y,z) + \propto yzh_y(x,y,0) \ge 0$, $yf_z(x,y,z) \ge 0$ for all x, y, z;
- $|p(t,x,y,z)| \le M + \delta_2(|y| + |z|)$

for all x, y, z and $t \ge 0$.

Then there exists a constant D whose magnitude depends on $a, b, c, \delta_0, \delta_1, \delta_2$ and M as well as the

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functions f, g and h such that every solution $(x(t), y(t), z(t))$ of the system (2) ultimately satisfies

$$|x(t)| \le D, |y(t)| \le D, |z(t)| \le D$$

for all $t \ge 0$.

For the rest of this article, D_1, D_2, D_3, . . . and the D's stand for positive constants. Their identities are preserved throughout this paper.

The proof of the theorem depends on the fundamental properties of the continuously differentiable function $V = V(x, y, z)$ defined as:

$$V = V_1 + V_2 \tag{3}$$

where

$$V_1 = \int_0^x h(\xi, 0,0)d\xi + \alpha \int_0^y g(x, \tau)d\tau + \alpha h(x, 0,0)y + \frac{\alpha}{2} z^2 + yz, \tag{4}$$

$$\frac{b}{c} > \alpha > \frac{1}{a} \tag{5}$$

and

$$V_2 = \begin{cases} x \,\mathrm{sgn}\, z, & \text{if } |z| \ge |x|, \\ z \,\mathrm{sgn}\, x, & \text{if } |z| \le |x|. \end{cases} \tag{6}$$

Observe that V_1 is a further generalization of the incomplete Lyapunov function $V(x, y, z)$ in [16], while V_2 is the same as a complete signum function in [5]. The required properties of V is contained in the following lemmas.

Lemma 1. Subject to the conditions of the theorem, there is a constant D_0 such that

$$V(x, y, z) > -D_0 \tag{7}$$

for all x, y, z and

$V(x, y, z) \to +\infty$ as $x^2 + y^2 + z^2 \to \infty$.

Proof. From (4), it is clear that V_1 can be rearrange in the form

$$V_1 = b^{-1} \int_0^x \left[b - \alpha h_\xi(\xi, 0,0)\right] h(\xi, 0.0)d\xi + \frac{\alpha}{2b}[by + h(x, 0,0)]^2 + \frac{1}{2\alpha}(\alpha z + y)^2$$
$$+\alpha \int_0^y [g(x, \tau) - b\tau]d\tau - \frac{\alpha}{2b} h^2(0,0,0)$$
$$+\alpha^{-1} \int_0^y [\alpha f(x, \tau, 0) - 1]\tau d\tau. \tag{8}$$

We now observe, view of hypothesis (i), that the integral

$$\alpha^{-1} \int_0^y [\alpha f(x, \tau, 0) - 1]\tau d\tau \ge \frac{1}{2\alpha}(\alpha a - 1)y^2 \tag{9}$$

Next, Chukwu's estimate in [5] shows quite clearly that

$$\alpha \int_0^y [g(x, \tau) - b\tau]d\tau \ge 0 \tag{10}$$

for $|y| \ge \Delta_0$, because the integrand satisfies $[g(x, \tau) - b\tau] sgn\, \tau \ge 0$ for $|\tau| \ge \Delta_0$. But if $|y| \le \Delta_0$, we have

$$\alpha \int_0^y [g(x, \tau) - b\tau]d\tau \ge -\Delta_0 \left[\delta_1 |h(x, 0, 0)| + \frac{b\Delta_0}{2}\right] \tag{11}$$

for all x and y.

Finally, in view hypothesis (iii)

$$b^{-1} \int_0^x \left[b - \alpha h_\xi(\xi, 0,0)\right] h(\xi, 0.0)d\xi \ge b^{-1}(b - \alpha c) \int_0^x h(\xi, 0,0)d\xi. \tag{12}$$

On combining (9), (10), (11) and (12), (8) becomes

$$V_1 \ge b^{-1}(b - \alpha c) \int_0^x h(\xi, 0,0)d\xi + \frac{1}{2\alpha}(\alpha a - 1)y^2 + \frac{1}{2\alpha}(\alpha z + y)^2 + \frac{\alpha}{2b}[by + h(x, 0,0)]^2 - \frac{\alpha}{2b} h^2(0,0,0) - \Delta_0 \left[\delta_1 |h(x, 0, 0)| + \frac{b\Delta_0}{2}\right] \tag{13}$$

Now, since α and b are positive constants it follows that

$$\frac{\alpha}{2b}[by + h(x, 0,0)]^2 \ge 0$$

for all x and y. Also by hypothesis (iv), for sufficiently large $|x|$, $|h(x, 0,0)| < h^2(x, 0,0)$. On gathering these estimates into (13), we obtain

$$V_1 \ge \frac{1}{2\alpha}(\alpha a - 1)y^2 + \frac{1}{2\alpha}(\alpha z + y)^2$$
$$+b^{-1}\left[b - \alpha a - 2bc\delta_1\Delta_0\right] \int_0^x h(\xi, 0,0)d\xi$$
$$-\frac{1}{2b}(\alpha + 2b\Delta_0\delta_1)h^2(0,0,0) - \frac{b\Delta_0^2}{2} \tag{14}$$

Since b is a positive constant, it follows by (5) that $b - \propto c > 0$, now assume that δ_1 is so small that

$$b - \alpha c > 2bc\delta_1\Delta_0, \tag{15}$$

there exist positive constants D_1, D_2, D_3 such that

$$V_1 \ge D_1 \int_0^x h(\xi, 0,0)d\xi + D_2 y^2 + \frac{1}{2\alpha}(\alpha z + y)^2 - D_3 \tag{16}$$

Finally, as in [5], [10] and [11], (6) yields

$$V_2 \ge -|z|. \tag{17}$$

On gathering the estimates (16) and (17) in (3), we get

$$V \ge D_1 \int_0^x h(\xi, 0,0)d\xi + D_2 y^2 + \frac{1}{2\alpha}(\alpha z + y)^2 - |z| - D_3 \tag{18}$$

In view of hypothesis (iv) of the theorem and the fact that $\propto$ is a positive constant Lemma 1 is immediate.

Lemma 2. Let (x(t), y(t), z(t)) be any solution of the differential system (2) and the function $V \equiv V(t)$ be defined by $V(t) \equiv V\big(t, x(t), y(t), z(t)\big)$. Then the limit

$$\dot{V}^\star(t) \equiv lim_{h\to 0^+} \frac{V[x(t+h), y(t+h), z(t+h)] - V[x(t), y(t), z(t)]}{h}$$

exist and there are positive constants D_4 and D_5 such that

$\dot{V}^\star(t) \le -D_4$ If $x^2 + y^2 + z^2 \ge D_5$.

Proof. Let $(x, y, z) \equiv (x(t), y(t), z(t))$ be any solution of (2). Also in [5], [10] and [13] the limit $\dot{V}^{\star}(t)$ corresponding to (x, y, z) clearly exists. An elementary calculation from (2) and (3) will infact yield the following equation

$$\dot{V}^{\star}(t) = \dot{V}_1 + \dot{V}_2 \quad (19)$$

Where

$$\begin{aligned}\dot{V}_1(t) = {} & y\int_0^y [\alpha g_x(x,\tau) + \tau f_x(x,\tau,0)]\,d\tau \\ & -(y+\alpha z)[h(x,y,z) - h(x,0,0)] \\ & -yz[f(x,y,z) - f(x,y,0)] \\ & -\left[\frac{g(x,y)}{y} - \alpha h_x(x,0,0)\right] y^2 \\ & -[\alpha f(x,y,z) - 1]z^2 \\ & +(y+\alpha z)p(t,x,y,z) \quad (20)\end{aligned}$$

and

$$\dot{V}_2(t) = \begin{cases} y\, sgn\, z & if\ |z| \geq |x|, \\ -[f(x,y,z)z + g(x,y) \\ \quad +h(x,y,z)]sgn\, x \\ \quad + p(t,x,y,z) sgn\, x & if\ |z| \leq |x|, \end{cases} \quad (21)$$

Observe that the first component in (20), in view of hypothesis (v) of the theorem, becomes

$$y\int_0^y [\propto g_x(x,\tau) + \tau f_x(x,\tau,0)]d\tau \leq 0 \quad (22)$$

for all x and y. The next two components in (20), by mean value theorem and assumption (vi) of the theorem, are non-negative. That is for all x, y, z

$$\begin{aligned}(y+\alpha z)[h(x,y,z) - h(x,0,0)] = {} & y^2 h_y(x,\theta_1 y,0) + \\ & \alpha z^2 h_z(x,y,\theta_2 z) + yz h_z(x,y,\theta_2 z) + \\ & \alpha yz\, h_y(x,\theta_1 y,0) \\ \geq {} & 0 \quad (23)\end{aligned}$$

where $0 \leq \theta_i \leq 1$ $(i = 1,2)$ and

$$yz[f(x,y,z) - f(x,y,0)] = z^2 y f_z(x,y,\theta_3) \geq 0 \quad (24)$$

where $0 \leq \theta_3 \leq 1$. On gathering estimates (22), (23) and (24), (20) becomes

$$\dot{V}_1 \leq -\left[\frac{g(xy)}{y} - \alpha h_x(x,0,0)\right] y^2 - [\alpha f(x,y,z) - 1]z^2 + (y+\alpha z)p(t,x,y,z). \quad (25)$$

From (21) and (25), we have the following estimates for (19)

$$\dot{V}^{\star}(t) \leq -W_0 + (y+\alpha z)p(t,x,y,z) + y sgn\, z \quad (26)$$

if $|z| \geq |x|$, and

$$\dot{V}^{\star}(t) \leq -[f(x,y,z)z + g(x,y) + h(x,y,z)]sgn\, x + (sgn\, x + y + \propto z)p(t,x,y,z) - W_0 \quad (27)$$

if $|z| \leq |x|$, where

$$W_{0\equiv} \left[\frac{g(x,y)}{y} - \alpha h_x(x,0,0)\right] y^2 + [\alpha f(x,y,z) - 1]z^2.$$

Let

$$W_1 = y\, sgn\, z + (y + \propto z)p(t,x,y,z) \quad (28)$$

if $|z| \geq |x|$ and

$$W_2 = (sgn\, x + y + \alpha z)p(t,x,y,z)$$

if $|z| \leq |x|$where W_1 and W_2 are group of terms in (26) and (27) respectively. Using the inequality $(|y| + |z|)^2 \leq 2(y^2 + z^2)$ and hypothesis (vii) of the theorem, we get

$$W_1 \leq D_7(|y| + |z|) + 2D_6\delta_2(y^2 + z^2) \quad (29)$$

and

$$W_2 \leq M + D_8(|y| + |z|) + 2D_6\delta_2(y^2 + z^2) \quad (30)$$

where $D_6 = max(\alpha, 1)$, $D_7 = max[(1 + D_6 M), D_6 M]$ and $D_8 = D_6 + \delta_2$. Let $\epsilon > 0$ be sufficiently small such that

$$\alpha > \frac{1}{a} + \epsilon. \quad (31)$$

Now set

$$W_3(x,y,z) \equiv Q_1 + Q_2$$

where

$$Q_1 = -[(\alpha - \epsilon)f(x,y,z) - 1]z^2 \quad (32)$$

And

$$Q_2 = -[\epsilon f(x,y,z)z^2 + f(x,y,z)zsgnx] + D_8|z| + 2\delta_2 D_6 z^2 \quad (33)$$

W_3 is contained partly in (27) and partly in (30). In view of hypothesis (i) of the theorem,

$$Q_1 \leq -[(\alpha - \epsilon)a - 1]z^2 \quad (34)$$

Since $(\propto - \epsilon)a > 1$ by (31), and

$$Q_2 \leq -(\epsilon\, \alpha - 2\delta_2 D_6)z^2 + (\alpha + D_8)|z| \quad (35)$$

provided δ_2 is chosen so small that $\epsilon a > 2\delta_2 D_6$.

An estimate for Q_2 will be calculated for the two cases when $|z|$ is arbitrarily large and when $|z|$ is bounded. When $|z|$ is arbitrarily large, say $|z| \geq D_9$, $0 < D_9 < \infty$ then $Q_2 \leq -1$ provided $\epsilon > 2\delta_2 D_6 a^{-1}$

On gathering the estimates Q_1 and Q_2 when $|z| \geq D_9$ and for all x, y, we have

$$W_3 \leq -[(\alpha - \epsilon)a - 1]z^2 - 1 \quad (36)$$

when $|z| \leq D_9$, $\epsilon < 2\delta_2 D_6 a^{-1}$ and for all x, y, we obtain from (33) that

$$\begin{aligned}Q_2 = {} & -[\epsilon f(x,y,z) - 2\delta_2 D_6]z^2 \\ & +[f(x,y,z) + D_8]|z| \\ \leq {} & -[\epsilon\alpha - 2\delta_2 D_6]z^2 \\ & +[\delta_0|h(x,y,z)| + D_8]|z| \\ \leq {} & [\delta_0|h(x,y,z)| + D_8]|z| \quad (37)\end{aligned}$$

where we have used the hypothesis (i) and the fact that $[\epsilon\alpha - 2\delta_2 D_6]\ z^2 \geq 0$
for all z if $\epsilon\alpha > 2\delta_2 D_6$ and δ_2 is chosen so small.

A combination of (33) and (37) when $|z| \leq D_9$ and for x, y yields

$$W_3 \leq [(\alpha - \epsilon)a - 1]z^2 + D_9[\delta_0|h(x,y,z)| + D_8] \quad (38)$$

Next, we let W_4 be group of terms contained partly in (27) and partly in (30), defined as

$$W_4 \equiv -\left[\left(\frac{g(x,y)}{y} - \alpha h_x(x,y,z)\right)y^2 + g(x,y)sgnx\right] + D_8|y| + 2\delta_2 D_6 y^2 + M \quad (39)$$

Again, the terms in W_4 will be estimated for two cases:

When $|y|$ is arbitrarily large, say, $|y| \geq D_{10}$ on applying hypotheses (ii) and (iii) of the theorem, we obtain

$$W_4 \leq -[(b - \alpha c) - 2\delta_2 D_6]y^2 + (b + D_8)|y| + M$$
$$\leq -[(b - \alpha c) - 2\delta_2 D_6]y^2 + D_{11}[|y| + 1]$$

where $D_{11} \equiv \max[b + D_8, M]$.

Choosing δ_2 so small that

$$b - \alpha c > 2\delta_2 D_6 \quad (40)$$

then

$$W_4 \leq -D_{12}y^2 + D_{11}(|y| + 1) \leq -1 \quad (41)$$

provided $|y| \geq D_{10}$ where $D_{12} \equiv [(b - \alpha c) - 2\delta_2 D_6] > 0$.

In the case $|y| \leq D_{10}$, we obtain

$$W_4 \leq -[(b - \alpha c) - 2\delta_2 D_6]y^2 + D_8 D_{10} + |g(x,y)| + M \leq D_{13}[max_{|y| \leq D_{10}}|g(x,y)| + 1] \leq D_{13}[|h(x,0,0)| + 1] \quad (42)$$

where $D_{13} \equiv \max[1, D_8 D_{10} + M]$, and since $D_{12}y^2 \geq 0$ for all y, provided that (40) holds.
Utilizing the estimates (36) and (41) in (27), we obtain

$$\dot{V}^*(t) \leq -h(x,y,x)sgnx - [(\alpha - \epsilon)a - 1]z^2 - 2 \quad (43)$$

for $|z| \geq D_9$, $|y| \geq D_{10}$ when $|x| \geq |z|$, provided that δ_2 is chosen so small so that $b - \alpha c > 2\delta_2 D_6$

On gathering the estimates (38) and (42) in (27), we have

$$\dot{V}^* \leq -h(x,y,x)sgnx - [(\alpha - \epsilon)a - 1]z^2 + D_9\big[\delta_0|h(x,y,z)| + D_{13}[\delta_1|h(x,0,0)| + 1]\big] \quad (44)$$

for $|z| \leq D_9$, $|y| \leq D_{10}$ when $|x| \geq |z|$.

Next, we find a suitable similar upper bound for (26) when $|z| \geq |x|$. To see this, let

$$W_5 \equiv -\left[\frac{g(x,y)}{y} - \alpha h_x(x,y,z)\right]y^2 + 2\delta_2 D_6 y^2 + D_7|y|$$

where W_5 the group of terms is partly contained in (26) and (29).

Now

$$W_5 \leq -[(b - \alpha c) - 2\delta_2 D_6]y^2 + D_7|y| \leq -D_{12}y^2 + D_7|y| \quad (45)$$

provided $|y| \geq D_{14}$, say.

In the case $|y| \leq D_{14}$, then

$$W_5 \leq -[((b - \alpha c) - 2\delta_2 D_6]y^2 + D_7 D_{14} \leq D_{15} \quad (46)$$

since $D_{12}y^2 \geq 0$, for all y and $D_{15} \equiv D_7 D_{14}$.

Finally, let W_6 be the group of terms partly contained in (26) and (29), defined as

$$W_6 \equiv -[\alpha f(x,y,z) - 1]z^2 + D_7|z| + 2\delta_2 D_6 z^2 \leq -D_{16}z^2 + D_7|z| \leq D \quad (47)$$

provided δ_2 is chosen so small that $\alpha a - 1 > 2\delta_2 D_6$.

Summing (45) and (47)
for, $|z| \geq |x|$ we obtain

$$\dot{V}^* \leq -D_{12}y^2 + D_7|y| + D \leq -1 \quad (48)$$

provided $|y| \geq D_{14}$ and $D_{12} > 0$. But in the event that $|y| \leq D_{14}$ the inequality in (26) together with (46) when $|z| \geq |x|$ yields

$$\dot{V}^* \leq -D_{16}z^2 + D_7|z| + D_{15} \leq -1 \quad (49)$$

provided $|z| \geq D_{17}$.

Proof of the Theorem:

As in [5], let $D_{18} \equiv \max[D_{17}, D_{14}, D_{10}, D_9]$. We now estimate (26) and (27) when:

- $y^2 + z^2 \geq 2D_{18}^2$ and $|x| \leq D_{18}$, that is when $y^2 + x^2$ is large in comparison with $|x|$;
- $|x|$ is large in comparison with $y^2 + z^2$, say $|x| \geq D_{18}$ but $y^2 + z^2 \leq 2D_{18}^2$. The final estimate will follow on combining (i) and (ii).

In the first case we estimate $\dot{V}^*$ in (43), (48) and (49). When $|z| \geq |x|$ and (43) is used, we obtain

$$\dot{V}^* \leq -2 - h(x,y,z)sgn\, x - [(\alpha - \epsilon)a - 1]z^2 \leq -1 \quad (50)$$

provided $|z| \geq D_{18}$ since $h(x,y,z)\,\mathrm{sgn}\, x \to +\infty$ as $|x| \to \infty$, then $-h(x,y,z)\,\mathrm{sgn}\, x = |h(x,y,z)| \leq D$ for all x, y, z.

When (48) and (49) are used, and $|z| \geq |x|$, we have

$\dot{V}^* \leq -1$ provided $y^2 + z^2 \geq 2D_{18}^2$ (51)

From (50) and (51) when $|z| \geq |x|$, we conclude that

$\dot{V}^* \leq -1$ whenever $y^2 + z^2 \geq 2D_{18}^2$ (52)

In the second case we consider $\dot{V}^*$ in (43) when $|x| \geq |z|$, we have

$$\dot{V}^* \leq -h(x,y,z)sgn\, x - [(\alpha - \epsilon)a - 1]z^2$$
$$+D_9[\delta_0|h(x,y,z)| + D_8] + D_{13}[\delta_1|h(x,0,0)| + 1]$$
$$\leq -[(1 - \delta_0 D_9)|h(x,y,z)| - \delta_1 D_{13}|h(x,0,0)|] + D_{19}$$

since $h(x,y,z)\,\mathrm{sgn}\, x = |h(x,y,z)|$ when $|x|$ is sufficiently large, $[(\alpha - \in)a - 1]z^2 \geq 0$ for all z provided $(\alpha - \in)a > 1$ and $D_{19} \equiv D_8 D_9 + D_{13}$.

If δ_0 is chosen so small that $1 > \delta_0 D_9$ and by hypothesis (iv) of the theorem, we obtain

$\dot{V}^* \leq -1$ provided $|x| \geq D_{20}$ (53)

but $|y|$ and $|z|$ are small. It is now clear from (52) and (53) that

$$\dot{V}^* \leq -1$$

whenever

$$x^2 + y^2 + z^2 \geq D_{20}^2 + 2D_{18}^2$$

This completes the proof of the Theorem.

Remark 1: Whenever $f(x,y,z) \equiv f(x,y)$ and $h(x,y,z) \equiv h(x)$, the hypotheses and conclusion of theorem coincide with those of Chukwu [5], except hypotheses (i) and (iii) of the theorem which are considerable weaker than those in [5]

Remark 2: If $f(x,\dot{x},\ddot{x}) \equiv \psi(\dot{x})$, $g(x,\dot{x}) \equiv \phi(x)\dot{x}$ and $p(t,x,\dot{x},\ddot{x}) \equiv p(t)$ then (1) reduces to the case studied by Ezeilo [10].

Remark 3: Suppose $f(x,y,z) \equiv a > 0$, where a is constant, $g(x,y) \equiv g(x)y, h(x,y,z) \equiv h(x)$ and $p(t,x,y,z) \equiv p(t)$, then the system (2) specializes to that investigated by Ezeilo and Tejumola in [13]. Moreover, the hypotheses and conclusion of the theorem coincide with those in [13], Theorem 1.

CONCLUSION

Obtaining ultimate boundedness of solutions for the nonlinear non-autonomous third order differential equation (1) using a complete Lyapunov function or the integral domain has remained difficult because it is computationally intensive. I n this paper, conditions obtained using a complete Yoshizawa functions (according to Chukwu [5]) are exact, see for instance [5], [10], and [13].

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