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NEW FORMULAS FOR ADJUSTING SHOOTING ANGLE IN APPLICATION OF SHOOTING METHOD TO BOUNDARY VALUE PROBLEMS

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Abstract: Two new formulas for adjusting shooting angle in the technique of shooting method were formulated. These formulas are modified Newton's formula and a Cubic Newton's formula which were obtained to form Taylor series. The effectiveness of these formulas were investigated by solving two different nonlinear boundary value problems, the results obtained were compared with the regular Newton's formula and both the new methods performed better that the Newton's method with the Cubic having the best performance. This inference is evident from the table of errors, tolerance value which is the absolute difference between two successive iterations also gave credence to the performance of the two formulas.

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1. Introduction

It interests scientists and engineers to model a system so as to understand the actions and behaviours of such system for prediction purposes. These behaviours for predictions cannot be possible without obtaining the solution of the model which are always differential equations in nature. In spite of the existence of standard methods for solving differential equations, numerical and semi-analytical methods have been the direction of solution in recent years. While Finite difference [1, 2, 3] and shooting methods [4, 5, 6] fall into the category of the early numerical methods, Adomian decomposition, homotopy analysis, differential transform and weighted residual methods [7, 8, 9] are examples of the semianalytical methods. In order to improve the level of accuracy and precision of each method, several modifications has been made to the aforementioned method. Shooting technique which this letter is centred is a method that transforms boundary value problems (BVP's) to initial value problems in which the resulting initial value problem are solved by any effective numerical method such as: Euler method, modified euler method, and Runge-Kutta method. It is conspicuous to also say that the order inherent in the boundary value problems determines the number of initial value problems to be solved. Among recent applications of shooting method is given by [11], which applied shooting technique to optimal control problems.

Furthermore, [10] applied operational matrix method to shooting technique to solve the resulting initial value problems. The hitch of the method under discussion for nonlinear differential equations is how to update the shooting angle or its slope of which the popular methods are Secant and Newton methods. The latter is found to be more accurate despite the two methods having quadratic convergence. In this write up, a modified shooting angle also of quadratic convergence and cubic order shooting angle is proposed and used to solve two point boundary value problems with shooting technique.

2. Methodology

Given a nonlinear differential equation:

$$y'' = f(x, y, y'), \quad a \leq x \leq \alpha, \quad y(a) = A, \quad y(\alpha) = B, \tag{1}$$

where a, α, A, B are constants and y', y'' are derivatives of y with respect to x for first and second order respectively.

By the procedure discussed in [1], equation (1) is reduced to

$$y'' = f(x, y, y'), \quad a \leq x \leq \alpha, \quad y(a) = A, \quad y'(a) = t = t_t, \tag{2}$$

Equation (2) involves parameter $t = t_k$, $k = 1, 2, 3, \cdots$, so that the shooting continues until

$$\lim_{k \to \infty} y(\alpha, t_k) = y(\alpha) = B.$$

Newton's method for determining the parameter t_k is also given by

$$t_k = t_{k-1} - \frac{y(\alpha, t_{k-1}) - B}{\frac{dy}{dx}(\alpha, t_{k-1})}. \tag{3}$$

Equation (3) was derived from the Newton's method of approximating nonlinear functions, which was also obtained from Taylor series. It is worthy to note that the explicit representation of $y(\alpha, t)$ is not known, hence the inclusion of $\frac{dy}{dt}(\alpha, t_{k-1})$ in equation (3) will be difficult to obtain, consequent upon this, an additional equation to generate $\frac{dy}{dt}$ is obtained from an assumed differential equations,

$$y''(x, t) = f(x,\ y(x, t),\ y'(x, t)), \quad a \leq x \leq \alpha, \tag{4}$$

$y(a, t) = A$, $y'(a, t) = t_k$ with parameter t such that:

$$\frac{\partial y''}{\partial t}(x, t) = \frac{\partial f}{\partial y}(x,\ y(x, t),\ y'(x, t))\frac{\partial y}{\partial t}(x, t)$$

$$+ \ \frac{\partial f}{\partial y}(x,\ y(x, t),\ y'(x, t))\frac{\partial y'}{\partial t}(x, t), \tag{5}$$

for $a \leq x \leq \alpha$ with the initial conditions

$$\frac{\partial y}{\partial t}(a, t) = 0, \qquad \frac{\partial y'}{\partial t}(a, t) = 1.$$

so if $\rho = \frac{\partial y}{\partial t}$, then equation (5) becomes;

$$\rho''(x, t) = f_y \rho + f_{y'} \rho', \quad \rho(a) = 0, \quad \rho'(a) = 1. \tag{6}$$

In practice, equations (2), (3), and (6) gives the approximation of equation (1) using Newton's method.

2.1. The modified shooting angle

Given the function $f(x) = 0$ which is nonlinear in nature, the root of the equation by Newton's formula is given by:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{7}$$

If $q(x) = \frac{f(x_i)}{f'(x_i)}$ then x_{i+1} can be written as

$$x_{i+1} = x_i - \frac{q(x_i)}{q'(x_i)}. \tag{8}$$

But

$$q'(x_i) = \frac{f'(x_i)f'(x_i) - f(x_i)f''(x_i)}{(f'(x_i))^2}. \tag{9}$$

Substituting equation (9) into equation (8) gives

$$x_{i+1} = x_i - \frac{f(x_i)f'(x_i)}{(f'(x_i))^2 - f(x_i)f''(x_i)}. \tag{10}$$

So if $t_k = x_{i+1}$ the equation (10) becomes;

$$t_k = t_{k-1} - \frac{f(t_{k-1})f'(t_{k-1})}{(f'(t_{k-1}))^2 - f(t_{k-1})f''(t_{k-1})}, \ k = 1, 2, 3, \cdots \tag{11}$$

The corresponding shooting angle to the equation then becomes

$$t_k = t_{k-1} - \left(\frac{(y(\alpha, t_{k-1}) - B)(\rho(\alpha, t_{k-1}))}{(\rho(\alpha, t_{k-1}))^2 - (y(\alpha, t_{k-1}) - B)(g(\alpha, t_{k-1}))} \right), \tag{12}$$

where $g = \frac{\partial^2 y}{\partial t^2}, \frac{\partial y}{\partial, t} = \rho$, $k = 1, 2, 3, \cdots$. Again, as $\rho(\alpha, t_{k-1})$ is not known, then equation (6) will be used to generate $\rho(\alpha, t_{k-1})$. It is also noteworthy that $g(\alpha, t_{k-1})$ will also be obtain as:

$$\begin{aligned} \frac{\partial^2 y''(x,t)}{\partial t^2} &= \frac{\partial^2 f}{\partial t^2}(x, \ y(x,y), \ y'(x,t)), \\ &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial z} \right), \\ &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial t} \right) \right) \end{aligned}$$

with $\frac{\partial x}{\partial t} = 0$,

$$\begin{aligned}\frac{\partial^2 y''(x,t)}{\partial t^2} =& \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\left(\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y \partial y'}\frac{\partial y'}{\partial t}\right) \\ &+ \frac{\partial f}{\partial y'}\frac{\partial^2 y'}{\partial t^2} + \frac{\partial y'}{\partial t}\left(\frac{\partial^2 f}{\partial y \partial y'}\frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y'}{\partial t}\right).\end{aligned}$$

If $\frac{\partial^2 y}{\partial t^2} = g$ and $\frac{\partial y}{\partial t} = \rho$, then

$$\begin{aligned}g''(x,t) =& \frac{\partial f}{\partial y}g + \rho\left(\frac{\partial^2 f}{\partial y^2}\rho + \frac{\partial^2 f}{\partial y \partial y'}\rho'\right) \\ &+ \frac{\partial f}{\partial y'}g' + \rho'\left(\frac{\partial^2 f}{\partial y \partial y'}\rho + \frac{\partial^2 f}{\partial (y')^2}\rho'\right)\end{aligned} \tag{13}$$

with the boundary $\frac{\partial}{\partial t}\rho'(a,t) = \frac{\partial}{\partial t}(1)$, this implies $g(a,t) = 0$. And $\frac{\partial}{\partial t}\rho' = 1$, this also implies $g'(a,t) = 0$.

So that the boundary conditions associated with equation (13) become:

$$g(a) = 0, \qquad g'(a) = 0.$$

Solution to equation (1) will now be approximated by solving equations (2), (6), and (13) subject to their corresponding boundary conditions while equation (12) is used to update or adjust the shooting angle. Any method of solving initial value problems could be employed to solve equations (2), (6), and (13) but Runge-Kutta method of order 4 is used in this write up.

2.2. Cubic Modified Shooting angle

While both Newton's method and the modified Newton's method converges quadratically to the actual solution, however in some cases, the Newton's method may not necessarily converge to the actual solution if the function is not a simple zero and this converges linearly to the solution.

Again, still considering

$$f(x) = 0, \tag{14}$$

expanding the left hand side of equation (14) by Taylor series gives;

$$f(x) + \frac{hf'(x)}{2!} + \cdots = 0,$$

where

$$h = \frac{-f(x) - \frac{h^2}{2}f''}{f'(x)},$$

h^2 will be obtained from the approximation $h \simeq \frac{-f(x)}{f'(x)}$, so that

$$h = \frac{\left(f(x) - \left(\frac{-f(x)}{f'(x)}\right)^2 f''(x).\frac{1}{2}\right)}{f'(x)}.$$

But $x_{i+1} = x_i + h$ and so,

$$x_{i+1} = x_i - \left(\frac{f(x)}{f'(x)} - \left(\left(\frac{f(x)}{f'(x)}\right)^2 \cdot \frac{f''(x)}{2f'(x)}\right)\right). \tag{15}$$

The corresponding shooting angle for equation (15) with $x_{i+1} = x_k$ is

$$t_k = t_{k-1} - \left(\frac{y(\alpha, t_{k-1}) - B}{\rho(\alpha, t_{k-1})} - \left(\frac{y(\alpha, t_{k-1}) - B}{\rho(\alpha, t_{k-1})}\right)^2 \cdot \left(\frac{g(\alpha, t_{k-1})}{2\rho(\alpha, t_{k-1}}\right)\right), \tag{16}$$

where $\frac{d}{dt}(B) = \frac{d^2}{dt^2}(B) = 0$.

Equation (16) is the cubic shooting angle which is used to update t_k until

$$y(\alpha, t_{k-1}) - B \simeq 0.$$

Equation for obtaining g has been generated in (13).

3. Numerical illustration

The modified quadratic shooting angle and the cubic alongside equation (13) are applied to solve two nonlinear boundary values problems and the results are compared with that of Newton's method of updating shooting angle.

Illustration 1: Consider the nonlinear differential equation

$$y'' = y^3 - yy', \quad 1 \leq x \leq 2, \quad y(1) = \frac{1}{2}, \quad y(2) = \frac{1}{3} \tag{17}$$

with exact solution $(x+1)^{-1}$.

Applying the shooting technique to equation (17), then

$$y'' = y^3 - yy', \quad 1 \leq x \leq 2, \quad y(1) = \frac{1}{2}, \quad y'(1) = t_k. \tag{18}$$

Here, $A = \frac{1}{2}$, $B = \frac{1}{3}$, $a = 1$, and $\alpha = 2$. So , $t_0 = \frac{1/3 - 1/2}{2-1} = \frac{-1}{6}$.

And

$$\rho''(x,t) = \frac{\partial f}{\partial y}\rho(x,t) + \frac{\partial f}{\partial y'}\rho'(x,t),$$

$$f(x,y,y') = y^3 - yy', \quad \frac{\partial f}{\partial y} = 3y^2 - y', \quad \frac{\partial f}{\partial y'} = -y.$$

So,

$$\rho' = (3y^2 - y')\rho - y\rho', \quad \rho(1) = 0, \quad \rho'(1) = 1. \tag{19}$$

And

$$g''(x,t) = \frac{\partial f}{\partial y}g + \rho\left(\frac{\partial^2 f}{\partial y^2}\rho + \frac{\partial^2 f}{\partial y \partial y'}\rho'\right) + \frac{\partial f}{\partial y'}g'$$

$$+ \rho'\left(\frac{\partial^2 f}{\partial y \partial y'}\rho + \frac{\partial^2 f}{\partial (y')^2}\rho'\right), \quad g(1) = 0, \quad g' = 0.$$

Now,

$$\frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial y \partial y'} = -1, \quad \frac{\partial^2 f}{(\partial y')^2} = 0.$$

So,

$$g'' = (3y^2 - y')g + \rho(6y\rho - \rho') - yg' + \rho'(-\rho + 0),$$

Then,

$$g'' = (3y^2 - y')g + \rho(6y\rho - \rho') - yg' - \rho\rho' \quad g(1) = 0, \quad g' = 0. \tag{20}$$

The systems of equations (18), (19), and (20) are solved repeatedly at different t_k using Runge-Kutta of order 4. Results at different values of t_k are shown with the aid of tables.

Illustration 2: Consider the nonlinear differential equation

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'), \quad 1 \le x \le 3, \quad y(1) = 17, \quad y(3) = \frac{43}{3} \tag{21}$$

having exact solution $= x^2 + \frac{16}{x}$.

Equation (21) becomes:

$$y'' = \frac{1}{8}(32 + 2x^3 - yy') \quad 1 \le x \le 3, \quad y(1) = 17, \quad y'(1) = t_k. \tag{22}$$

Here, $A = 17$, $B = \frac{43}{3}$, $a = 1$, and $\alpha = 3$. Then, $t_0 = \frac{\frac{43}{3} - 17}{3-1} = \frac{-4}{3}$.

And

$$\rho''(x,t) = \frac{\partial f}{\partial y}(x,t) + \frac{\partial f}{\partial y'}(x,t), \tag{23}$$

$$f(x,y,y') = \frac{1}{8}(32 + 2x^3 - yy'), \quad \frac{\partial f}{\partial y} = \frac{-y'}{8}, \quad \frac{\partial f}{\partial y'} = \frac{-y}{8}.$$

So,

$$\rho''(x,t) = \frac{-y'}{8}\rho - \frac{-y}{8}\rho' = \frac{-1}{8}(y'\rho + y\rho'), \quad \rho(1) = 0, \quad \rho'(1) = 1. \tag{24}$$

Also,

$$g''(x,t) = \frac{\partial f}{\partial y}g + \rho\left(\frac{\partial^2 f}{\partial y^2}\rho + \frac{\partial^2 f}{\partial y \partial y'}\rho'\right) + \frac{\partial f}{\partial y'}g'$$
$$+ \rho'\left(\frac{\partial^2 f}{\partial y \partial y'}\rho + \frac{\partial^2 f}{\partial (y')^2}\rho'\right), \quad g(1) = 0, \quad g' = 0.$$

Now,

$$\frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial y \partial y'} = \frac{-1}{8}, \quad \frac{\partial^2 f}{(\partial y')^2} = 0.$$

Then,

$$g'' = \frac{-1}{8}y'g - \frac{-1}{8}\rho\rho' - \frac{-1}{8}yg' - \frac{-1}{8}\rho\rho'.$$

And so,

$$g'' = \frac{-1}{8}(gy' + yg' + 2\rho\rho'). \tag{25}$$

Equations (22), (24), and (25) will be solved iteratively as well.

4. Results

Problem 2

Table 1: Errors obtained from the three methods at $t_0 = \frac{-1}{6}$ for problem 1

x	Newton's method	modified Newton's method	modified cubic method
1.0	0	0	0
1.2	1.597E-03	1.59E-02	1.59E-02
1.4	3.10E-03	3.10E-02	3.10E-02
1.6	4.56E-02	4.56E-02	4.56E-02
1.8	6.02E-02	6.02E-02	6.01E-02
2.0	7.48E-02	7.48E-02	7.48E-02

Table 2: Errors obtained from the three methods at different t_1 of the method for problem 1

x	Newton's method $t_1 = -0.250349639340$	modified Newton's method $t_1 = -0.249702603226$	modified cubic method $t_1 = -0.250023596153$
1.0	0	0	0
1.2	6.70E-05	5.70E-05	4.52E-05
1.4	1.30E-04	1.11E-04	8.79E-06
1.6	1.92E-04	1.63E-04	1.29E-05
1.8	2.53E-04	2.15E-04	1.71E-05
2.0	3.15E-04	2.68E-04	2.13E-05

5. Discussion of results

Equation (12) is the modified Newton's formula for finding shooting angle which is quadratic with respect to accuracy while equation (16) is the cubic formula also for finding shooting angle. Owing to a second order derivative represented by g included in both equations (12) and (16) the equation (13) was as well calculated.

Tables 1, 2, and 3 show the error obtained for Newton's formula, modified Newton's formula, and the cubic formula at different shooting angles and it

Table 3: Errors obtained from the three methods at different t_2 of the method for problem 1

x	Newton's method $t_2 = -0.250000012167$	modified Newton's method $t_2 = -0.250000000347$	modified cubic method $t_2 = -0.250000000348$
1.0	0	0	0
1.2	1.95E-09	8.79E-10	9.90E-11
1.4	2.46E-09	2.00E-09	9.80E-11
1.6	5.46E-10	2.78E-09	1.90E-11
1.8	4.78E-09	4.48E-09	7.89E-10
2.0	6.35E-09	4.61E-09	1.00E-12

Table 4: Errors obtained from the three methods at different t_3 of the method for problem 1

x	Newton's method $t_3 = -0.250000005130$	modified Newton's method $t_3 = -0.250000000347$	modified cubic method $t_3 = -0.250000000347$
1.0	0	0	0
1.2	6.06E-10	8.79E-10	9.90E-11
1.4	1.60E-10	2.00E-09	9.80E-11
1.6	3.32E-09	2.78E-09	1.90E-11
1.8	3.32E-10	4.48E-09	7.89E-10
2.0	0	4.61E-09	1.00E-12

was observed that the cubic formula has the least error followed by modified Newton's formula.

Table 4 shows the tolerance of the shooting angle at each stage of computation. Tolerance is the absolute difference between two successive iteration and the smaller it becomes the closer the methods tends to the solution. It was observed that at second iteration, the modified Newton's and cubic formula already tends to zero while it took the Newton's formula up till the fourth iteration to achieve this value.

Table 5: Absolute difference between two successive shooting angle for the three methods for problem 1

Tolerance	Newton's method	modified Newton's method	modified cubic method
$\|t_1 - t0\|$	8.36E-02	8.30E-02	8.33E-02
$\|t_2 - t1\|$	3.49E-04	2.97E-04	2.35E-04
$\|t_3 - t2\|$	7.10E-09	0	0
$\|t_4 - t3\|$	0	0	0

Table 6: Errors obtained from the three methods at $t_0 = \frac{-4}{3}$ for problem 2

x	Newton's method	modified Newton's method	modified cubic method
1.0	0	0	0
1.4	3.5210	3.5210	3.5210
1.8	5.2542	5.2542	5.2542
2.2	6.0611	6.0611	6.0611
2.6	6.2911	6.2911	6.2911
3.0	6.1458	6.1458	6.1458

Tables 5, 6, 7, and 8 show the error for each method on problem 2 and similar trend of performance was also observed in all the tables as cubic gives the least error and Newton's method gave the maximum error. In addition, the tolerance associated with that problem could be found in Table 9 where cubic already tends to zero at 4th iteration, modified Newton's method tends to zero at 5th while it took 6th iteration for Newton's method to a zero tolerance.

6. Conclusions

A modified Newton's formula and Cubic formula to update shooting angle while applying shooting technique have been derived. The performance of the two

Table 7: Errors obtained from the three methods at different t_1 of the method for problem 2

x	Newton's method $t_1 = -16.3174867248$	modified Newton's method $t_1 = -12.6883415993$	modified cubic method $t_1 = -13.9229632702$
1.0	0	0	0
1.4	3.06665	0.3471	2.20E-02
1.8	1.0606	0.5852	3.46E-02
2.2	1.3048	0.7065	4.21E-02
2.6	1.4229	0.7581	4.54E-02
3.0	1.4278	0.7527	4.52E-02

Table 8: Errors obtained from the three methods at different t_2 of the method for problem 2

x	Newton's method $t_2 = -14.1085694940$	modified Newton's method $t_2 = -13.9705482217$	modified cubic method $t_2 = -13.9999996397$
1.0	0	0	0
1.4	3.11E-02	8.42E-03	5.59E-08
1.8	4.89E-02	1.32E-02	2.06E-08
2.2	5.95E-02	1.61E-02	2.86E-08
2.6	6.42E-02	1.73E-02	1.99E-08
3.0	6.39E-02	1.73E-02	3.60E-08

formulas is compared with the usual Newton's formula and the results show that the Cubic formula gives the best performance followed by the modified Newton's formula.

Tolerance at each shooting angle also gives credence to the performance of the new methods. It is therefore recommended that this two formulas can be applied whenever shooting technique is used for quick and better convergence of the solution.

Table 9: Errors obtained from the three methods at different t_3 of the method for problem 2

x	Newton's method $t_3 = -14.0002254590$	modified Newton's method $t_3 = -13.9999831737$	modified cubic method $t_3 = -13.9999997009$
1.0	0	0	0
1.4	36.46E-05	4.65E-06	7.34E-08
1.8	1.02E-04	7.43E-06	7.0E-09
2.2	1.23E-04	9.04E-06	4.90E-09
2.6	1.33E-04	9.73E-06	1.62E-08
3.0	1.33E-04	9.71E-06	0

Table 10: Errors obtained from the three methods at different t_4 of the method for problem 2

x	Newton's method $t_4 = -13.9999997018$	modified Newton's method $t_4 = -13.9999997009$	modified cubic method $t_4 = -13.9999997009$
1.0	0	0	0
1.4	7.36E-08	7.34E-08	7.34E-08
1.8	7.40E-09	7.0E-09	7.0E-09
2.2	5.40E-09	4.90E-09	4.90E-09
2.6	1.67E-08	1.62E-08	1.62E-08
3.0	5.0E-10	0	0

Table 11: Errors obtained from the three methods at different t_5 of the method for problem 2

x	Newton's method $t_5 = -13.9999997009$	modified Newton's method $t_5 = -13.9999997009$	modified cubic method $t_5 = -13.9999997009$
1.0	0	0	0
1.4	7.34E-08	7.34E-08	7.34E-08
1.8	7.0E-09	7.0E-09	7.0E-09
2.2	4.90E-09	4.90E-09	4.90E-09
2.6	1.62E-08	1.62E-08	1.62E-08
3.0	0	0	0

Table 12: Absolute difference between two successive shooting angle for the three methods for problem 2

Tolerance	Newton's method	modified Newton's method	modified cubic method
$\|t_1 - t_0\|$	14.98415339	11.35500827	12.58912994
$\|t_2 - t_1\|$	2.20891723	1.2822066224	7.70E-02
$\|t_3 - t_2\|$	1.08E-01	2.29E-02	6.12E-08
$\|t_4 - t_3\|$	2.25E-04	1.67E-05	0
$\|t_5 - t_4\|$	9.0E-10	0	0
$\|t_6 - t_5\|$	0	0	0

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