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Efficient Numerical Methods for Solving Nonlinear High-Order Boundary Value Problems

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Abstract

Nonlinear problems are prevalent in plasma physics, solid state physics, fluid dynamics, chemical kinetics, structural and continuum mechanics and there is high demand for computational tools to solve these problems. In this paper, we have developed two methods, namely, Embedded Perturbed Chebyshev Integral Collocation Method and Embedded Perturbed Bernstein Integral Collocation Method for the efficient numerical solution of nonlinear high-order boundary value problems. Newton-Raphson-Kantorovich linearization scheme of appropriate orders are employed to linearize the nonlinear equations and then resorting to iterative procedure. Numerical results are included to demonstrate the reliability and efficiency of the methods. Comparison between exact and approximate solutions is made and our proposed methods compared favourably with known numerical methods used in solving the same nonlinear problems considered.

Keywords: *Bernstein polynomial, Chebyshev polynomial, Boundary value problem, Collocation method, Nonlinear problem.*

1 Introduction

Mathematical modeling of many physical systems leads to nonlinear ordinary differential equations. An effective method is required to analyze the mathematical modeling which provides solution conforming to physical reality [1]. In the literature, many analytical and numerical methods have been studied for solution of nonlinear problems. Since solution of nonlinear boundary value problems in an analytical form is difficult, often times numerical methods have been used to solve these types of problems.

Many authors have used numerical and approximate methods to solve fourth-order boundary value problems. Details about some of these numerical methods are found in the recent references like Al-Hayani and Casasus [2], Songxin Liang and David [3], Geng [4] and Olagunju [5]. Fifth-order boundary value problems arise in the mathematical modeling of viscoelastic flows and other branches of science and engineering which have been widely studied by many authors. Recently, Ghazala and Homoodur in [6] presented solution to the fifth-order boundary value problems in reproducing kernel space, Ayyaz et al. [7] applied Hermite wavelets method for solving this class of problems. In 2007 El-Gamel [8] employed the Sinc-Galerkin method and Shaowei [9] applied homotopy perturbation method to address the numerical issues related to this type of problem.

On the contrary, the literature on the numerical solution of sixth-order boundary value problems is sparse. Such problems are known to arise in astrophysics, the narrow convicting layers bounded by stable layers [10]. Fazal-i-Haq et al. [11] applied collocation method using Haar wavelets. Chebyshev Pseudospectral method for the solutions of sixth-order boundary value problems was used by El-Kady and Khalil [12] whereas Adomian decomposition, homotopy perturbation and variational iteration methods were used to investigate solutions of sixth-order boundary value problems by Wazwaz [13], Noor and Mohyud-Din [14] and Noor et al.[15], respectively.
Thus, we consider the following general class of nonlinear differential equation of order p:

$$y^p(x) = f(x, y(x), y'(x), ..., y^{p-1}(x)) \tag{1}$$

Subject to the two-point boundary conditions

$$\begin{gathered} y(a) = \alpha_0, y'(a) = \alpha_1, ..., y^r(a) = \alpha_r, \\ y(b) = \beta_0, y'(b) = \beta_1, ..., y^{p-r-2}(b) = \beta_{p-r-2} \end{gathered} \tag{2}$$

where $0 \le r \le p-2$ is an integer and $a, b, \alpha_0, \alpha_1, ...\alpha_r, \beta_0, \beta_1, ..., \beta_{p-r-2}$ are real constants.

2 Basis Functions

2.1 Bernstein Polynomials

The general form of Bernstein Polynomials of the *mth* degree on the interval $[a,b]$ is defined by

$$B_{i,m}(x) = \binom{m}{i} \frac{(x-a)^i (b-x)^{m-i}}{(b-a)^m}, i = 0, 1, ..., m \tag{3}$$

. The **B**-polynomials are generated by a recursive relation

$$B_{i,m}(x) = \frac{b-x}{b-a} B_{i,m-1}(x) + \frac{x}{b-a} B_{i-1,m-1}(x) \tag{4}$$

2.2 Chebyshev Polynomials of the First Kind $T_N(x)$

The Chebyshev Polynomials of the first kind on the interval $[-1,1]$ and with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ is defined by the trigonometric identity:

$$T_n(x) = \cos(n \cos^{-1} x) \tag{5}$$

and the recurrence relation is defined as

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1, 2, \cdots \tag{6}$$

These could be converted into any interval $[a,b]$ to obtain

$$T_n(x) = \cos\left[n \cos^{-1}\left(\frac{2x-a-b}{b-a}\right)\right], \tag{7}$$

and the recursive relation is given as

$$T_{n+1}(x) = 2\left\{\frac{2x-a-b}{b-a}\right\} T_n(x) - T_{n-1}(x) \tag{8}$$

3 Description of Solution Techniques

In this section, we use Chebyshev and Berstein polynomials as basis functions for developing two numerical methods for the solution of equation (1) and (2).

3.1 Method 1: Embedded Perturbed Chebyshev Integral Collocation Method (EPCICM)

In this method, the approximate scheme of the p-th order derivative is firstly sought in truncated Chebyshev series form with perturbation term added and then integrated p-times to obtain expressions for lower-order derivatives and the function itself. Thus, the process is described as follows:

$$\frac{d^p y_{N,k+1}(x)}{dx^p} = \sum_{n=0}^{N} a_n T_n(x) + \chi_v H_N(x) \tag{9}$$

Integrating equation (9) successively, we obtain

$$\begin{aligned} \frac{d^{p-1} y_{N,k+1}(x)}{dx^{p-1}} &= \sum_{n=0}^{N} a_n \int T_n(x)dx + \chi_v \int H_N(x)dx + c_1 \\ &= \sum_{n=0}^{N+1} \delta_{n,p-1} \phi_n^{[p-1]}(x) + \chi_v \psi^{[p-1]}(x) \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{d^{p-2} y_{N,k+1}(x)}{dx^{p-2}} &= \sum_{n=0}^{N} a_n \int \phi_n^{[p-1]}(x)dx + \chi_v \int \psi^{[p-1]}(x)dx + c_1 x + c_2 \\ &= \sum_{n=0}^{N+2} \delta_{n,p-2} \phi_n^{[p-2]}(x) + \chi_v \psi^{[p-2]}(x) \end{aligned} \tag{11}$$

$$\vdots$$

$$\begin{aligned} \frac{d y_{N,k+1}(x)}{dx} &= \sum_{n=0}^{N+1} a_n \int \phi_n^{[2]}(x)dx + \chi_v \int \psi^{[2]}(x)dx + c_1 \frac{x^{p-2}}{(p-2)!} \\ &\quad + c_2 \frac{x^{p-3}}{(p-3)!} + \cdots + c_{p-2} x + c_{p-1} \\ &= \sum_{n=0}^{N+p-1} \delta_{n,1} \phi_n^{[1]}(x) + \chi_v \psi^{[1]}(x) \end{aligned} \tag{12}$$

$$\begin{aligned} y_{N,k+1}(x) &= \sum_{n=0}^{N} a_n \int \phi_n^{[1]}(x)dx + \chi_v \int \psi^{[1]}(x)dx + c_1 \frac{x^{p-1}}{(p-2)!} \\ &\quad + c_2 \frac{x^{p-2}}{(p-2)!} + \cdots + c_{p-1} x + c_p \\ &= \sum_{n=0}^{N+p} \delta_{n,0} \phi_n^{[0]}(x) + \chi_v \psi^{[0]}(x) \end{aligned} \tag{13}$$

where

$$\chi_v = \begin{cases} 1, & v = p \\ 0, & v \neq p \end{cases}, \tag{14}$$

$$H_n(x) = \tau_1 T_N(x) + \tau_2 T_{N-1}(x) + \tau_3 T_{N-2}(x) + \cdots + \tau_n T_{N-n+1}(x) \tag{15}$$

3.2 Method 2: Embedded Perturbed Bernstein Integral Collocation Method (EPBICM)

Bernstein approximations are adopted here to approximate the solution of linearized problem. Starting with Bernstein approximations for the highest order derivative, we generate approximations to the lowest-order derivatives as follows:

$$\frac{d^p y_{m,k+1}(x)}{dx^p} = \sum_{i=0}^{m} c_i B_{i,m}(x) + \chi_v H_N(x) \tag{16}$$

Successive integration of equation (3.2.1) gives

$$\begin{aligned}\frac{d^{p-1} y_{m,k+1}(x)}{dx^{p-1}} &= \sum_{i=0}^{m} c_i \int B_{i,m}(x)dx + \chi_v \int H_N(x)dx + k_1 \\ &= \sum_{i=0}^{m+1} \zeta_{i,p-1}\Phi_i^{[p-1]}(x) + \chi_v \psi^{[p-1]}(x)\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{d^{p-2} y_{m,k+1}(x)}{dx^{p-2}} &= \sum_{i=0}^{m} c_i \int \Phi_i^{[p-1]}(x)dx + \chi_v \int \psi^{[p-1]}(x)dx + k_1 x + k_2 \\ &= \sum_{i=0}^{m+2} \zeta_{i,p-2}\Phi_i^{[p-2]}(x) + \chi_v \psi^{[p-2]}(x)\end{aligned} \tag{18}$$

$$\vdots$$

$$\begin{aligned}\frac{d y_{m,k+1}(x)}{dx} &= \sum_{i=0}^{m} c_i \int \Phi_i^{[2]}(x)dx + \chi_v \int \psi^{[2]}(x)dx + k_1 \frac{x^{p-1}}{(p-2)!} \\ &+ k_2 \frac{x^{p-3}}{(p-3)!} + \cdots + k_{p-2}x + k_{p-1} \\ &= \sum_{i=0}^{m+p-1} \zeta_{i,1}\Phi_i^{[1]}(x) + \chi_v \psi^{[1]}(x)\end{aligned} \tag{19}$$

$$\begin{aligned}y_{m,k+1}(x) &= \sum_{i=0}^{m} c_i \int \Phi_i^{[1]}(x)dx + \chi_v \int \psi^{[1]}(x)dx + k_1 \frac{x^{p-1}}{(p-2)!} \\ &+ k_2 \frac{x^{p-2}}{(p-2)!} + \cdots + k_{p-1}x + k_p \\ &= \sum_{i=0}^{m+p} \zeta_{i,0}\Phi_i^{[0]}(x) + \chi_v \psi^{[0]}(x)\end{aligned} \tag{20}$$

4 Application of the Methods on Some Examples

In this section, we apply Embedded Perturbed Chebyshev Integral Collocation and Embedded Perturbed Bernstein Integral Collocation Methods to the following three examples to study the efficiency of these new methods.

Example 1: Consider the nonlinear boundary value problem

$$y^{iv}(x) = \sin x + \sin^2 x - \left(y''(x)\right)^2, 0 < x < 1 \tag{21}$$

subject to

$$y(0) = 0, \quad y'(0) = 1, \quad y(1) = \sin(1), \quad y'(1) = \cos(1) \tag{22}$$

The exact solution is $y = \sin x$.

Method 1: EPCICM
Linearizing (21) and (22) by Newton's Linearization scheme, we obtain

$$y^{iv}_{N,k+1}(x) + 2y''_{N,k}y''_{N,k+1} - \left(y''_{N,k}\right)^2 = \sin x + (1 - \cos 2x)/2 \tag{23}$$

together with the boundary conditions:

$$y_{N,k+1}(0) = 0, y'_{N,k+1}(0) = 1, y_{N,k+1}(1) = \sin(1), y'_{N,k+1}(1) = \cos(1) \tag{24}$$

We rewrite (23) and (24) as

$$\left(\sum_{n=0}^{N} a_n T_n(x) + H_N(x)\right) + 2\left(\sum_{n=0}^{N+2} \delta_{n,2}\phi_n^{[2]}(x) + \right) y''_k(x) - (y''_k)^2 = \sin x + \frac{1}{2}(1 - \cos 2x) \tag{25}$$

together with the boundary conditions

$$\sum_{n=0}^{N+4} \delta_{n,0}\phi_n^{[0]}(0) = 1, \quad \sum_{n=0}^{N+3} \delta_{n,1}\phi_n^{[1]}(0) = 1$$

$$\sum_{n=0}^{N+4} \delta_{n,0}\phi_n^{[0]}(1) = \sin(1), \quad \sum_{n=0}^{N+3} \delta_{n,1}\phi_n^{[1]}(1) = \cos(1) \tag{26}$$

Equation (25) together with the boundary conditions (26) are solved for various values of N. For $N = 6$ and $k = 0$, equation (25) becomes

$$\left(\sum_{n=0}^{6} a_n T_n(x) + H_6(x)\right) + 2\left(y_0''(x)\right)\left(\sum_{n=0}^{8} \delta_{n,2}\phi_n^{[2]}(x) + \right) - \left(y_0''\right)^2 = \sin x + \frac{1}{2}(1-\cos 2x) \quad (27)$$

In order to solve equation (27), we use the initial approximation

$$y_{6,0} = 0.166666666667x + x^3 + 0.008330627300x^5 - 0.0001929758000x^7 \quad (28)$$

Substituting equation (28) into equation (27) and collocating at the point $x_i = \frac{i}{12},\ i = 1, 2, \cdots, 11$, obtained from

$$x_i = a + \frac{(b-a)i}{N+p+2}, \qquad i = 1, 2, \cdots, N+p+1, \quad (29)$$

where in this case $a = 0$, $b = 1$, $N = 6$ and $p = 4$, we obtain a linear system of algebraic equations which were solved by using Gaussian elimination method. Thus, we obtained the following approximate solution at the second iteration (i.e $k = 1$):

$$y_{6,2}(x) = x - 0.00002073882400x^2 - 0.1666218330x^3 + 0.00007677560x^4$$
$$+0.008058248323x^5 + 0.0002526313625x^6 - 0.00028987863037x^7$$
$$+0.00001666997805x^8 + 0.000000158755089x^9 + 0.000001048802006x^{10} \quad (30)$$

Method 2: EPBICM

Similarly, equations (21) and (22) can be written as

$$\left(\sum_{i=0}^{m} c_n B_{i,m}(x) + H_N(x)\right) + 2\left(\sum_{i=0}^{m+2} \zeta_{i,2}\phi_i^{[2]}(x)\right) y_k''(x) - \left(y_k''(x)\right)^2 = \sin x + \frac{1}{2}(1-\cos 2x) \quad (31)$$

subject to the boundary conditions

$$\sum_{i=0}^{m+4} \zeta_{i,0}\phi_i^{[0]}(0) = 0, \qquad \sum_{i=0}^{m+3} \zeta_{i,1}\phi_i^{[1]}(0) = 1,$$
$$\sum_{i=0}^{m+4} \zeta_{i,0}\phi_i^{[0]}(0) = \sin(1), \qquad \sum_{i=0}^{m+3} \zeta_{i,1}\phi_i^{[1]}(1) = \cos(1) \quad (32)$$

Also, equation (31) together with the boundary conditions (32) were solved for various values of m. For $m = 6$ and $k = 0$, equation (31) becomes

$$\left(\sum_{i=0}^{6} c_i B_{i,m}(x) + H_6(x)\right) + 2\left(\sum_{i=0}^{8} \zeta_{i,2}\phi_i^{[2]}(x)\right) y_0''(x) - (y_0''(x))^2 = \sin x + \frac{1}{2}(1-\cos 2x) \tag{33}$$

Similarly, we take (28) as our the initial approximation. Substituting equation (28) into equation (33) and collocating the resulting equation at the point $x_i = \frac{i}{12}$, $i = 1, 2, \cdots, 11$, obtained from

$$x_i = a + \frac{(b-a)i}{m+p+2}, \qquad i = 1, 2, \cdots, m+p+1, \tag{34}$$

where in this case $a = 0$, $b = 1$, $m = 6$ and $p = 4$, we obtain a linear system of algebraic equations which were also solved using Guassian elimination method. Thus, the following approximate solution is obtained after the second iteration:

$$y_{6,2}(x) = x + 0.000009451606150x^2 - 0.1666867022x^3 - 0.00003336257465x^4$$
$$+0.008449387170x^5 - 0.00010035824x^6 - 0.00016748488x^7 - 0.00000342220x^8$$
$$+0.000002815863x^9 + 0.0000006601977x^{10} \tag{35}$$

In Table 1, we compare the EPCICM and EPBICM solutions with the results given in Kasi [16].

Table 1: Comparison of Absolute Errors for Example 1

x	Exact Solution	Kasi [16]	EPCICM	EPBICM
0.1	9.983342E-02	9.685755E-07	1.5739E-07	7.2200E-08
0.2	1.986693E-01	4.082918E-06	4.2110E-07	1.9550E-07
0.3	2.955202E-01	7.957220E-06	5.3740E-07	2.5409E-07
0.4	3.894183E-01	1.019239E-05	4.0530E-07	2.0180E-07
0.5	4.794255E-01	1.180172E-05	8.6600E-08	6.0800E-08
0.6	5.646425E-01	1.358986E-05	2.5150E-07	9.3800E-08
0.7	6.442177E-01	1.221895E-05	4.3260E-07	1.8190E-07
0.8	7.173561E-01	7.748604E-06	3.7120E-07	1.6120E-07
0.9	7.833269E-01	4.529953E-06	1.4560E-06	6.4300E-08

Example 2: Consider the following nonlinear fifth-order boundary value problem.

$$y^{v}(x) + y^{iv}(x) + e^{-2x}y^2(x) = 2e^x + 1, \quad 0 \le x \le 1 \tag{36}$$

with the boundary conditions

$$y(0) = 1,\ y(1) = e,\ y'(0) = 1,\quad y''(0) = 1,\ y'(1) = e \tag{37}$$

The exact solution is $y(x) = e^x$

Following similar procedures as in Example 1 and using the initial approximation

$$y_{10,0}(x) = 1 + x + \frac{1}{2}x^2 + 0.1548454840x^3 + 0.06343634400x^4,$$

we obtain for EPCICM and EBCICM respectively:

$$y_{10,2}(x) = 1+x+\frac{1}{2}\,x^2+0.1665003953\,x^3+0.04190521488\,x^4+0.008521212263\,x^5 -0.002197683039\,x^6+0.001649000348\,x^7-0.03845304254\,x^8+0.05533702161\,x^9 -0.05137388927\,x^{10}+0.03113117471\,x^{11}-0.01198364856\,x^{12}+0.002670654273\,x^{13} -0.0002659592912\,x^{14} \tag{38}$$

and

$$y_{10,2}(x) = 1+x+\frac{1}{2}\,x^2+0.1672174867\,x^3+0.04066460878\,x^4+0.008193034328\,x^5 -0.00167656697\,x^6+0.02963066603\,x^7-0.05567466273\,x^8+0.00753999683\,x^9 +0.0637162762\,x^{10}-0.02552924823\,x^{11}-0.05717413626\,x^{12}+0.05629798326\,x^{13} -0.01491679445\,x^{14}-0.0000068148921\,x^{15} \tag{39}$$

Table 2: Comparison of Numerical Results for Example 2

x	Ghazala [6]	Shaowei [9]	El-Gamel [8]	EPCICM	EPBICM
0.0000	0.0000000	0.000000	0.000000	0.0000000	0.0000000
0.0100	7.790E-10	1.20E-09	0.000000	0.0000000	0.0000000
0.1184	7.830E-07	7.00E-06	0.000000	2.310E-07	7.130E-07
0.1517	1.360E-06	1.40E-05	1.00E-04	4.620E-07	1.383E-06
0.2410	3.005E-06	4.60E-05	0.000000	1.620E-06	4.429E-06
0.3604	2.520E-06	1.00E-04	1.00E-04	4.424E-06	1.130E-05
0.4287	2.570E-07	1.00E-04	0.000000	6.460E-06	1.665E-05
0.5000	5.040E-06	1.90E-04	2.00E-04	8.581E-06	2.279E-05
0.6395	1.580E-06	1.00E-04	1.00E-04	1.121E-05	3.049E-05
0.8482	1.330E-05	9.80E-05	2.00E-04	6.422E-06	1.303E-05
0.9996	2.100E-10	1.20E-09	2.00E-04	0.0000000	0.0000000
1.0000	0.0000000	0.000000	0.000000	0.0000000	1.000E-09

Example 3: Consider the following nonlinear sixth-order boundary value problem [8]:

$$y^{vi}(x) + e^{-x}y^2(x) = e^{-x} + e^{-3x}, \quad x \in [0, 1] \tag{40}$$

with the boundary conditions

$$y(0) = 1, \quad y(1) = \frac{1}{e}, \quad y'(0) = -1, \quad y'(1) = -\frac{1}{e}, y''(0) = 1, \quad y''(1) = \frac{1}{e} \tag{41}$$

The exact solution is $y(x) = e^{-x}$

In this case, we choose our initial approximation to be

$$y_{10,0}(x) = 1 + x + \frac{1}{2}x^2 - 12.16574810x^3 + 16.03877286x^4 - 6.005145310x^5 \tag{42}$$

Thus, we obtain the following approximate solutions for EPCICM and EP-BICM when $k = 2$, respectively:

$$\begin{aligned} y_{10,3}(x) &= 1 - x + \frac{1}{2}x^2 - 0.1666674725\,x^3 + 0.04166827808\,x^4 - 0.008333240203\,x^5 \\ &+ 0.001388177179\,x^6 - 0.0001400572767\,x^7 - 0.0003260168284\,x^8 + 0.0008750754469\,x^9 \\ &- 0.001199687135\,x^{10} + 0.0009631179874\,x^{11} - 0.0004399414459\,x^{12} + 0.00009280475233\,x^{13} \\ &+ 0.00000033017516\,x^{14} - 0.0000004491774777\,x^{16} - 0.000001477869152\,x^{15} \end{aligned} \tag{43}$$

and

$$\begin{aligned} y_{10,3}(x) &= 1 - x + \frac{1}{2}x^2 - 0.1662194345\,x^3 + 0.04061060740\,x^4 - 0.007938270464\,x^5 \\ &+ 0.0003918767422\,x^6 + 0.005733246040\,x^7 - 0.009311105185\,x^8 + 0.00333904281\,x^9 \\ &+ 0.01274607440\,x^{10} - 0.03199312714\,x^{11} + 0.04194495636\,x^{12} - 0.03608331681\,x^{13} \\ &+ 0.02096506126\,x^{14} - 0.007576413256\,x^{15} + 0.001270243529\,x^{16} \end{aligned} \tag{44}$$

Table 3: Numerical Results and Errors for Example 3 when $k = 2$ (Third iteration)

Method	x	Exact Solution	Approximate Solution	Error
EPCICM	0.0	1.0000000000	1.0000000000	0.0000000000
EPBICM			1.0000000000	0.0000000000
EPCICM	0.1	0.9048374180	0.9048374174	6.0000E-10
EPBICM			0.9048377640	3.4600E-07
EPCICM	0.2	0.8187307531	0.8187307493	3.8000E-09
EPBICM			0.8187327590	2.0059E-06
EPCICM	0.3	0.7408182207	0.7408182131	7.6000E-09
EPBICM			0.7408122762	4.5413E-06
EPCICM	0.4	0.6703200460	0.6703200373	8.7000E-09
EPBICM			0.6703265760	6.5300E-06
EPCICM	0.5	0.6065306597	0.6065306538	5.9000E-09
EPBICM			0.6065374560	6.7963E-06
EPCICM	0.6	0.5488116361	0.5488116341	2.0000E-09
EPBICM			0.5488168970	5.2604E-06
EPCICM	0.7	0.4965853038	0.4965853038	0.00000000
EPBICM			0.4965882350	2.9312E-06
EPCICM	0.8	0.4493289641	0.4493289641	0.00000000
EPBICM			0.4493300160	1.0519E-06
EPCICM	0.9	0.4065696597	0.4065696595	2.0000E-10
EPBICM			0.4065698190	1.5930E-07
EPCICM	1.0	0.3678794412	0.3678794413	1.2856E-10
EPBICM			0.3678794420	8.2855E-10

5 Conclusion

In this work, we have used Embedded Perturbed Integral Collocation Methods for finding the solutions of fourth, fifth and sixth-orders nonlinear boundary value problems. In these methods expressions for the dependent variable (y) and its the derivatives are explicitly defined and no restrictive assumptions are needed. The solutions obtained show excellent agreement with the exact solutions as remarkably low errors are notable.

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