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On the Laplace Homotopy Analysis Method for a Nonlinear System of Second-Order Boundary Value Problems

O.M. Ogunlaran[1] and A.T. Ademola[2]

[1]Department of Mathematics and Statistics
Bowen University, Iwo, Nigeria
E-mail: dothew2002@yahoo.com
[2]Department of Mathematics
University of Ibadan, Ibadan, Nigeria
E-mail: ademola672000@yahoo.com

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Abstract

In this paper, we present an efficient algorithm for solving system of second-order boundary value problems(BVPs) based on a combination of the Laplace transform and homotopy analysis method(HAM). The proposed technique finds the solution without any discretization or any restrictive assumptions. The solution is produces in form of a rapid convergent series. The results of the numerical examples considered show that the proposed method is applicable and efficient.

Keywords: *Homotopy analysis method, Laplace transform, nonlinear System of equations, Boundary value problems, Numerical methods.*

1 Introduction

In recent years, investigations of systems of boundary value problems have been the focus of many studies due to their wide range of applicability in physics, engineering, biology and other fields [3]. However, many classical methods used to solve second-order initial value problems cannot be applied to solve second-order boundary value problems. For a nonlinear system of second-order

boundary value problems, there are few valid methods to obtain numerical solution [1, 3, 6]. Therefore, attention has been paid to searching for better and more efficient methods for determining a solution, approximate or exact, analytical or numerical, to the system of second-order differential equations. Among the methods that have been developed for handling a system of second-order boundary value problems are B-spline method [4], Variation iteration method [3], Sinc-collocation method [16], Chebyshev finite difference method [6], spline collocation approach [5], and homotopy perturbation method(HPM) [7].
In this study, we propose a coupled method of Laplace transform and homotopy analysis method to obtain the solutions of the following nonlinear system of second-order ordinary differential equations [3].

$$u_1'' + a_1(t)u_1' + a_2(t)u_1 + a_3(t)u_2'' + a_4(t)u_2' + a_5(t)u_2 + G_1(t, u_1, u_2) = f_1(t)$$

$$u_2'' + b_1(t)u_2' + b_2(t)u_2 + b_3(t)u_1'' + b_4(t)u_1' + b_5(t)u_1 + G_2(t, u_1, u_2) = f_2(t) \quad (1)$$

subject to the boundary conditions

$$u_1(0) = 0, \quad u_1(1) = 0, \quad u_2(0) = 0 \quad \text{and} \quad u_2(1) = 0 \qquad (2)$$

where $0 \le t \le 1$, G_1 and G_2 are nonlinear functions of u_1 and u_2. $a_i(t)$, $b_i(t)$, $f_1(t)$ and $f_2(t)$ are given functions, and $a_i(t)$, $b_i(t)$, $i = 0, 1, \cdots 5$ are continuous.

The Laplace transform is a wonderful tool for solving linear differential equations and has enjoyed much success at this level. However, it is totally inapplicable to nonlinear equations because of the difficulties caused by nonlinear terms. Since Laplace Adomian decomposition method (LADM) was proposed by Khuri [17] and then developed by Khan [18] and Khan and Gondal [19], the couple methods that are based on Laplace transform and other methods have received considerable attention in the literature. For instance, in [20, 22, 23, 24, 25, 26, 27] and in [21] the homotopy perturbation method and the variational iteration method are combined with the well-known Laplace transform to develop a highly effective technique for handling many nonlinear problems.

The homotopy analysis method initially proposed by Liao [2], is based on homotopy, a fundamental concept in topology and differential geometry. By means of HAM, one construct a continuous mapping of an initial approximation to the exact solution of the given equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure the convergence of series solution. The method enjoys great freedom in choosing initial approximation and auxiliary linear operators. HAM is an efficient method for solving various form of both linear and nonlinear problems [2]. Recently, this method has been successfully applied to solve different types of nonlinear problems in science and engineering [8, 9, 10, 11, 12, 13, 14, 15]. All these successful applications verified the validity, effectiveness and flexibility of the homotopy analysis method.

2 Laplace Homotopy Analysis Method

Consider the system of second-order differential equations [7]

$$u_i''(t) = f_i(t, u_1, u_2, u_1', u_2'), i = 1, 2 \tag{3}$$

subject to the boundary conditions

$$u_i(0) = 0, \qquad u_i(1) = 0 \tag{4}$$

By applying the Laplace transform, denoted in this paper by $\mathfrak{L}$, on both sides of (3), we get

$$\mathfrak{L}\left\{u_i''(t)\right\} = \mathfrak{L}\left\{f_i(t, u_1, u_2, u_1', u_2')\right\}, i = 1, 2 \tag{5}$$

Using the differentiation property of the Laplace transform on the LHS, we have

$$s^2\mathfrak{L}\left\{u_i(t)\right\} - u_i'(0) - su_i(0) = \mathfrak{L}\left\{f_i(t, u_1, u_2, u_1', u_2')\right\} \tag{6}$$

$$\text{Now, let} \qquad u_i'(0) = c_i \tag{7}$$

On simplifying, we obtain

$$\mathfrak{L}\left\{u_i(t)\right\} - \frac{c_i}{s^2} - \frac{1}{s^2}\mathfrak{L}\left\{f_i(t, u_1, u_2, u_1', u_2')\right\} = 0, i = 1, 2 \tag{8}$$

The system of nonlinear operators are given as

$$\begin{aligned} N_i\left[\phi_i(t;q)\right] &= \mathfrak{L}\left\{\phi_i(t;q)\right\} - \frac{c_i}{s^2} \\ &-\frac{1}{s^2}\mathfrak{L}\left\{[f_i(t, \phi_1(t;q), \phi_2(t;q), \frac{\partial \phi_1(t;q)}{\partial t}, \frac{\partial \phi_2(t;q)}{\partial t})\right\}, i = 1, 2 \end{aligned} \tag{9}$$

where $\phi_i(t;q)$ are real functions of t and q.

We construct a homotopy as follows:

$$(1-q)\mathfrak{L}\left\{\phi_i(t;q) - u_{i,0}(t)\right\} = q\hbar_i N_i\left[\phi_i(t;q)\right], i = 1, 2 \tag{10}$$

where $q \in [0, 1]$ is an embedding parameter, $\hbar_i$ are nonzero auxiliary parameters, $u_{i,0}(t)$ are initial guesses of $u_i(t)$ and $\phi_i(t;q)$ are unknown functions. Obviously, when $q = 0$ and $q = 1$ we have

$$\phi_i(t;0) = u_{i,0}(t) \qquad \text{and} \qquad \phi_i(t;1) = u_i(t) \tag{11}$$

Thus, as q increases from 0 to 1, the solution $\phi_i(t;q)$ vary from the initial guesses $u_{i,0}(t)$ to the solutions $u_i(t)$.
Expanding $\phi_i(t;q)$ in Taylor series with respect to q, we get

$$\phi_i(t;q) = u_{i,0}(t) + \sum_{m=1}^{\infty} u_{i,m}(t)q^m \tag{12}$$

where

$$u_{i,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t;q)}{\partial q^m}|_{q=0} \tag{13}$$

If the auxiliary the auxiliary parameters h_i and the initial guesses are properly chosen such that the power series (12) converge at $q = 1$ then we have under these conditions the series solutions

$$u_i(t) = u_{i,0}(t) + \sum_{m=1}^{\infty} u_{i,m}(t) \tag{14}$$

The governing equations can be deduced from the zeroth-order deformation equations (10).

Define the vectors

$$\vec{u}_{i,n} = \{u_{i,0}(t), u_{i,1}(t), u_{i,2}(t), \cdots, u_{i,n}(t)\} \tag{15}$$

Differentiating the zeroth-order deformation equations (10) m times with respect to the embedding parameter q, dividing by $m!$ and finally setting $q = 0$, we obtain the mth-order deformation equations:

$$\mathfrak{L}\{u_{i,m}(t) - \chi_m u_{i,m-1}(t)\} = \hbar_i R_{i,m}(\vec{u}_{i,m-1}) \tag{16}$$

Applying the inverse Laplace transform, we have

$$u_{i,m}(t) = \chi_m u_{i,m-1}(t) + \hbar_i \mathfrak{L}^{-1}\{R_{i,m}(\vec{u}_{i,m-1})\} \tag{17}$$

where

$$R_{i,m}(\vec{u}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\{N_i[\phi_i(t;q)]\}|_{q=0}, i = 1,2 \tag{18}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{19}$$

3 Illustrative Examples

In this section, we apply the proposed Laplace homotopy analysis method to solve some systems of second-order boundary value problems in order to establish the applicability and the accuracy of the method.

Example 3.1 *Consider the linear system of boundary value problems [3, 7]:*

$$\begin{cases} u_1''(t) + tu_1(t) + tu_2(t) = g_1(t) \\ u_2''(t) + 2tu_2(t) + 2tu_1(t) = g_2(t) \qquad 0 \leq t \leq 1 \end{cases} \tag{20}$$

subject to the boundary conditions

$$u_1(0) = u_2(0) = 0, \quad u_1(1) = u_2(1) = 0 \tag{21}$$

where $g_1(t) = 2$ and $g_2(t) = -2$
The exact solutions of this problem are $u_1(t) = t^2 - t$ and $u_2(t) = t - t^2$
In view of the boundary conditions (21), we determine the initial guesses as

$$u_{i,0}(t) = 0 \tag{22}$$

Taking Laplace transform on both sides of system (20) subject to the initial conditions we have

$$\mathfrak{L}\{u_1(t)\} - \frac{c_1}{s^2} + \frac{1}{s^2}\mathfrak{L}\{tu_1(t)\} + \frac{1}{s^2}\mathfrak{L}\{tu_2(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_1(t)\} = 0 \tag{23}$$

$$\mathfrak{L}\{u_2(t)\} - \frac{c_2}{s^2} + \frac{2}{s^2}\mathfrak{L}\{tu_1(t)\} + \frac{2}{s^2}\mathfrak{L}\{tu_2(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_2(t)\} = 0 \tag{24}$$

The system of nonlinear operators is defined as

$$N_1[\phi_1(t;q)] = \mathfrak{L}\{\phi_1(t;q)\} - \frac{c_1}{s^2} + \frac{1}{s^2}\mathfrak{L}\{t\phi_1(t;q)\} + \frac{1}{s^2}\mathfrak{L}\{t\phi_2(t;q)\} - \frac{1}{s^2}\mathfrak{L}\{g_1(t)\} \tag{25}$$

$$N_2[\phi_2(t;q)] = \mathfrak{L}\{\phi_2(t;q)\} - \frac{c_2}{s^2} + \frac{2}{s^2}\mathfrak{L}\{t\phi_1(t;q)\} + \frac{2}{s^2}\mathfrak{L}\{t\phi_2(t;q)\} - \frac{1}{s^2}\mathfrak{L}\{g_2(t)\} \tag{26}$$

and thus

$$\begin{aligned} R_{1,m}(\vec{u}_{1,m-1}) = \mathfrak{L}\{u_{1,m-1}(t)\} - \frac{c_1}{s^2} + \frac{1}{s^2}\mathfrak{L}\{tu_{1,m-1}(t)\} \\ + \frac{1}{s^2}\mathfrak{L}\{tu_{2,m-1}(t)\} - \frac{1}{s^2}(1-\chi_m)\mathfrak{L}\{g_1(t)\} \end{aligned} \tag{27}$$

$$R_{2,m}(\vec{u}_{2,m-1}) = \mathfrak{L}\{u_{2,m-1}(t)\} - \frac{c_2}{s^2} + \frac{2}{s^2}\mathfrak{L}\{tu_{1,m-1}(t)\} + \frac{2}{s^2}\mathfrak{L}\{tu_{2,m-1}(t)\} - \frac{1}{s^2}(1-\chi_m)\mathfrak{L}\{g_2(t)\} \quad (28)$$

The mth-order deformation equation with $\hbar = -1$ is given by

$$\mathfrak{L}\{u_{i,m}(t) - \chi_m u_{i,m-1}(t)\} = -R_{i,m}(\vec{u}_{i,m-1}) \quad (29)$$

Taking the inverse Laplace transform, we get

$$u_{i,m}(t) = \chi_m u_{i,m-1}(t) - \mathfrak{L}^{-1}\{R_{i,m}(\vec{u}_{i,m-1})\} \qquad m = 1, 2, \cdots \quad (30)$$

where the constants c_i $(i = 1, 2)$ in equations (30) are determined by applying the conditions (21)b.
Now by defining $U_{i,n}(t) = \sum_{k=0}^{n} u_{i,k}(t)$, $i = 1, 2$ we obtain
$U_{1,1}(t) = t^2 - t$ and $U_{2,1}(t) = t - t^2$
which are the exact solutions to this problem.

Example 3.2 *Consider the following linear system of second-order boundary value problems [3, 7]:*

$$\begin{cases} u_1''(t) + (2t-1)u_1'(t) + \cos(\pi t)u_2'(t) = g_1(t) \\ u_2''(t) + tu_1(t) = g_2(t) \qquad 0 \le t \le 1 \end{cases} \quad (31)$$

subject to the boundary conditions

$$u_1(0) = u_2(0) = 0, \quad u_1(1) = u_2(1) = 0 \quad (32)$$

where $g_1(t) = -\pi^2 \sin(\pi t) + (2t-1)\pi\cos(\pi t) + (2t-1)\cos(\pi t)$ and
$g_2(t) = 2 + t\sin(\pi t)$
The exact solutions of this problem are $u_1(t) = \sin(\pi t)$ and $u_2(t) = t^2 - t$
We obtain the initial guesses from the boundary conditions (32) as

$$u_{i,0}(t) = 0 \quad (33)$$

Taking Laplace transform on both sides of system (31) subject to the initial conditions we have

$$\mathfrak{L}\{u_1(t)\} - \frac{c_1}{s^2} + \frac{2}{s^2}\mathfrak{L}\{tu_1'(t)\} - \frac{1}{s^2}\mathfrak{L}\{u_1'(t)\} + \frac{1}{s^2}\mathfrak{L}\{\cos(\pi t)u_2'(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_1(t)\} = 0 \quad (34)$$

$$\mathfrak{L}\{u_2(t)\} - \frac{c_2}{s^2} + \frac{1}{s^2}\mathfrak{L}\{tu_1(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_2(t)\} = 0 \quad (35)$$

The system of nonlinear operators is defined as

$$N_1[\phi_1(t;q)] = \mathfrak{L}\{\phi_1(t;q)\} - \frac{c_1}{s^2} + \frac{2}{s^2}\mathfrak{L}\left\{t\frac{\partial\phi_1(t;q)}{\partial t}\right\} - \frac{1}{s^2}\mathfrak{L}\left\{\frac{\partial\phi_1(t;q)}{\partial t}\right\}$$

$$+\frac{1}{s^2}\mathfrak{L}\left\{\cos(\pi t)\frac{\partial\phi_2(t;q)}{\partial t}\right\} - \frac{1}{s^2}\mathfrak{L}\{g_1(t)\} \quad (36)$$

$$N_2[\phi_2(t;q)] = \mathfrak{L}\{\phi_2(t;q)\} - \frac{c_2}{s^2} + \frac{1}{s^2}\mathfrak{L}\{t\phi_1(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_2(t)\} \quad (37)$$

Thus,

$$R_{1,m}(\vec{u}_{1,m-1}) = \mathfrak{L}\{u_{1,m-1}(t)\} - \frac{c_1}{s^2} + \frac{2}{s^2}\mathfrak{L}\left\{tu'_{1,m-1}(t)\right\} - \frac{1}{s^2}\mathfrak{L}\left\{u'_{1,m-1}(t)\right\}$$

$$+\frac{1}{s^2}\mathfrak{L}\left\{\cos(\pi t)u'_{2,m-1}(t)\right\} - \frac{1}{s^2}\mathfrak{L}\{g_1(t)\}(1-\chi_m) \quad (38)$$

$$R_{2,m}(\vec{u}_{2,m-1}) = \mathfrak{L}\{u_{2,m-1}(t)\} - \frac{c_2}{s^2} + \frac{1}{s^2}\mathfrak{L}\{tu_{2,m-1}(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_2(t)\}(1-\chi_m) \quad (39)$$

The mth-order deformation equation with $\hbar = -1$ is given by

$$L\mathfrak{L}\{u_{i,m}(t) - \chi_m u_{i,m-1}(t)\} = -R_{i,m}(\vec{u}_{i,m-1}) \quad (40)$$

Taking the inverse Laplace transform, we get

$$u_{i,m}(t) = \chi_m u_{i,m-1}(t) - \mathfrak{L}^{-1}\{R_{i,m}(\vec{u}_{i,m-1})\} \qquad m = 1, 2, \cdots \quad (41)$$

where the constants c_i $(i = 1, 2)$in equations (41) are determined by applying the conditions (32)b.
Thus, the first-order terms are
$u_{1,1}(t) = -\frac{1}{\pi^2}(2t(\pi+1) - \pi - 1)\cos(\pi t) + \frac{1}{\pi^3}(\pi^3 + 4\pi + 4)\sin(\pi t) - \frac{\pi+1}{\pi^2}$
$u_{2,1}(t) = -\frac{2\cos(\pi t)}{\pi^3} - \frac{t\sin(\pi t)}{\pi^2} + t^2 - t\left(\frac{4}{\pi^3} + 1\right) + \frac{2}{\pi^3}$
$u_{i,m}(t), i = 1, 2, m = 2, 3, \cdots$ can be calculated in the same manner.
Now we define the maximum absolute errors for $U_{i,n}$ as
$EU_{i,n} = max\,|U_{i,n}(t_k) - u_i(t_k)|$, $t_k = 0.1k$, for $k = 0(1)10$, $i = 1, 2$
Table 3.1 shows that the errors decrease as the value of n increases.

Example 3.3 *In this example, we consider the nonlinear system:*

$$\begin{cases} u''_1(t) - tu'_2(x) + u_1(t) = g_1(t) \\ u''_2(t) + tu'_1(t) + u_1(t)u_2(t) = g_2(t), \qquad 0 \le t \le 1 \end{cases} \quad (42)$$

subject to the boundary conditions

$$u_1(0) = u_2(0) = 0, \quad u_1(1) = u_2(1) = 0 \tag{43}$$

where $g_1(t) = t^3 - 2t^2 + 6t$ and $g_2(t) = t^5 - t^4 + 2t^3 + t^2 - t + 2$
The exact solutions of this problem are $u_1(t) = t^3 - t$ and $u_2(t) = t^2 - t$
For this problem, we take the initial guesses as

$$u_{i,0}(t) = 0 \tag{44}$$

By applying the Laplace homotopy analysis method to (42) subject to the initial conditions, we have

$$\mathfrak{L}\{u_1(t)\} - \frac{c_1}{s^2} - \frac{1}{s^2}\mathfrak{L}\{tu_2'(t)\} + \frac{1}{s^2}\mathfrak{L}\{u_1(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_1(t)\} = 0 \tag{45}$$

$$\mathfrak{L}\{u_2(t)\} - \frac{c_2}{s^2} + \frac{1}{s^2}\mathfrak{L}\{tu_1'(t)\} + \frac{1}{s^2}\mathfrak{L}\{u_1(t)u_2(t)\} - \frac{1}{s^2}\mathfrak{L}\{g_2(t)\} = 0 \tag{46}$$

The system of nonlinear operators is

$$N_1[\phi_1(t;q)] = \mathfrak{L}\{\phi_1(t;q)\} - \frac{c_1}{s^2} - \frac{1}{s^2}\mathfrak{L}\left\{t\frac{\partial\phi_2(t;q)}{\partial t}\right\} + \frac{1}{s^2}\mathfrak{L}\{\phi_1(t;q)\} - \frac{1}{s^2}\mathfrak{L}\{g_1(t)\} \tag{47}$$

$$N_2[\phi_2(t;q)] = \mathfrak{L}\{\phi_2(t;q)\} - \frac{c_2}{s^2} + \frac{1}{s^2}\mathfrak{L}\left\{t\frac{\partial\phi_1(t;q)}{\partial t}\right\} + \frac{1}{s^2}\mathfrak{L}\{\phi_1(t;q)\phi_2(t;q)\} - \frac{1}{s^2}\mathfrak{L}\{g_2(t)\} \tag{48}$$

and thus

$$R_{1,m}(\vec{u}_{1,m-1}) = \mathfrak{L}\{u_{1,m-1}(t)\} - \frac{c_1}{s^2} - \frac{1}{s^2}\mathfrak{L}\{tu_{2,m-1}'(t)\} + \frac{1}{s^2}\mathfrak{L}\{u_{1,m-1}(t)\} - \frac{1}{s^2}(1-\chi_m)\mathfrak{L}\{g_1(t)\} \tag{49}$$

$$R_{2,m}(\vec{u}_{2,m-1}) = \mathfrak{L}\{u_{2,m-1}(t)\} - \frac{c_2}{s^2} + \frac{1}{s^2}\mathfrak{L}\{tu_{1,m-1}'(t)\} + \frac{1}{s^2}\mathfrak{L}\left\{\sum_{i=0}^{m-1} u_{1,i}(t)u_{2,m-1-i}(t)\right\} - \frac{1}{s^2}(1-\chi_m)\mathfrak{L}\{g_2(t)\} \tag{50}$$

The $mth-$order deformation equation with $\hbar = -1$ is given by

$$\mathfrak{L}\{u_{i,m}(t) - \chi_m u_{i,m-1}(t)\} = -R_{i,m}(\vec{u}_{i,m-1}) \tag{51}$$

Applying the inverse Laplace transform, we have

$$u_{i,m}(t) = \chi_m u_{i,m-1}(t) - \mathfrak{L}^{-1}\{R_{i,m}(\vec{u}_{i,m-1})\} \tag{52}$$

where the constants c_i $(i = 1, 2)$ in equations (52) are determined by applying the conditions (43)b.
Thus solving (52), for $m = 1, 2, \cdots$, we get

$$u_{1,1}(t) = \frac{t^5}{20} - \frac{t^4}{6} + t^3 - \frac{53}{60}t$$

$$u_{2,1}(t) = \frac{t^7}{42} - \frac{t^6}{30} + \frac{t^5}{10} + \frac{t^4}{12} - \frac{t^3}{6} + t^2 - \frac{141}{140}t$$

$$u_{1,2}(t) = \frac{t}{15120}(35t^8 - 54t^7 + 162t^6 + 252t^5 - 1134t^4 + 2520t^3 - 312t^2 - 1469)$$

$$u_{2,2}(t) = -\tfrac{t}{2520}(15t^6 - 56t^5 + 378t^4 - 371t^2 + 34)$$

$$\vdots$$

The absolute errors $|U_{i,n}(t) - u_i(t)|$ for different values of t and n are tabulated in Tables 3.2 and 3.3

Table 3.1: Maximum absolute errors for Example 3.2

n	1	3	5	7
$EU_{1,n}$	1.1×10^{-1}	1.6×10^{-3}	2.2×10^{-5}	3.0×10^{-7}
$EU_{2,n}$	5.1×10^{-2}	8.3×10^{-4}	1.1×10^{-5}	1.6×10^{-7}

Table 3.2: Absolute errors $|U_{1,n}(x) - u_1(x)|$ for Example 3.3

x	$n = 1$	$n = 3$	$n = 5$	$n = 7$
0.1	1.2×10^{-2}	2.6×10^{-4}	1.3×10^{-6}	6.0×10^{-8}
0.2	2.3×10^{-2}	4.9×10^{-4}	2.7×10^{-6}	1.3×10^{-7}
0.3	3.4×10^{-2}	6.3×10^{-4}	4.2×10^{-6}	2.2×10^{-7}
0.4	4.3×10^{-2}	6.7×10^{-4}	6.0×10^{-6}	3.3×10^{-7}
0.5	4.9×10^{-2}	5.9×10^{-4}	7.7×10^{-6}	4.6×10^{-7}
0.6	5.2×10^{-2}	3.9×10^{-4}	9.2×10^{-6}	5.8×10^{-7}
0.7	5.0×10^{-2}	1.3×10^{-4}	9.9×10^{-6}	6.4×10^{-7}
0.8	4.1×10^{-2}	1.3×10^{-4}	9.2×10^{-6}	5.9×10^{-7}
0.9	2.5×10^{-2}	2.3×10^{-4}	6.3×10^{-6}	3.7×10^{-7}

Table 3.3: Absolute errors $|U_{2,n}(x) - u_2(x)|$ for Example 3.3

x	$n = 1$	$n = 3$	$n = 5$	$n = 7$
0.1	8.7×10^{-4}	1.9×10^{-4}	1.4×10^{-5}	2.9×10^{-7}
0.2	2.6×10^{-3}	3.9×10^{-4}	2.7×10^{-5}	5.8×10^{-7}
0.3	5.7×10^{-3}	6.0×10^{-4}	3.9×10^{-5}	8.4×10^{-7}
0.4	1.0×10^{-2}	7.8×10^{-4}	4.9×10^{-5}	1.0×10^{-6}
0.5	1.6×10^{-2}	9.2×10^{-4}	5.5×10^{-5}	1.2×10^{-6}
0.6	2.3×10^{-2}	9.9×10^{-4}	5.5×10^{-5}	1.2×10^{-6}
0.7	2.7×10^{-2}	9.7×10^{-4}	4.9×10^{-5}	1.2×10^{-6}
0.8	2.8×10^{-2}	8.0×10^{-4}	3.7×10^{-5}	9.9×10^{-7}
0.9	2.1×10^{-2}	4.7×10^{-4}	2.1×10^{-5}	6.2×10^{-7}

4 Conclusion

We have introduced a coupled method of Laplace transform and homotopy analysis method (HAM) for the solution of a nonlinear system of second-order boundary problems. This approach does not involve discretization unlike traditional techniques used by other numerical algorithms. The solution is given in a series form which converges rapidly. It is also worth mentioning that the method is applied without any linearization or restrictive assumptions being made. The method is simple and highly accurate as evident from the results of the numerical examples considered

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