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# SPLINE SOLUTION METHODS FOR A CLASS OF SINGULAR BOUNDARY VALUE PROBLEMS

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## Abstract

We present some methods based on a non-polynomial spline function for the numerical solution of a certain class of singular two-point boundary value problems. The methods involve a modification of the singular boundary value problem considered at the singular point. The methods are tested on some examples and the numerical results are compared with the exact solutions and some other methods in literatures.

## 1. Introduction

Consider the singular two-point boundary value problem of the form:

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$$u''(x) + \frac{k}{x}u'(x) + b(x)u(x) = c(x), \quad 0 \leq x \leq 1, \tag{1}$$

$$u'(0) = 0, \quad u(1) = \beta, \tag{2}$$

where $k \geq 1$ and $\beta$ is a constant.

These problems occur in the study of generalized axially symmetric potentials after separation of variables. Such problems also arise frequently in areas such as thermal explosion, electrohydrodynamics, chemical reactions and, atomic and nuclear physics [5-7]. Singular boundary value problems have been studied by several authors. Jamet [9] discussed the existence and uniqueness of solutions and presented a finite difference method for solving such problems. Erikson and Thomee [8] described Galerkin methods for this class of singular boundary value problems. Bickley [10] studied the use of cubic spline for solving linear two-point boundary value problems. Taiwo and Ogunlaran in [11, 13] considered solving fourth-order boundary value problems by polynomial and non-polynomial spline methods, respectively. Kanth and Reddy [1, 3] described a fourth-order finite difference method and a cubic approximation method, respectively, for this class of singular boundary value problem. Caglar and Caglar [2] presented a B-spline method for solving a system of differential equations, while Goh et al. [12] presented an extended uniform B-spline method of third degree for solving singular boundary value problem.

The series expansion procedure is the common technique for removing the singularity. Kanth and Reddy [4] used the Chebyshev economization near the singularity on $(0, \delta)$ and solved the regular boundary value problem in the interval $(\delta, 1)$ by using a central difference method.

In this paper, we describe non-polynomial spline methods for solving a class of singular two-point boundary value problems. The original differential equation is modified at the singular point by the application of the L'Hopital rule. Examples of both homogeneous and non-homogeneous singular boundary value problems are solved to demonstrate the efficiency and accuracy of the methods.

## 2. Description of the Methods

As a result of singularity at $x = 0$, the differential equation (1) is modified at the singular point in order to remove the singularity by applying L'Hopital rule. Hence, the boundary value problem (1) with (2) transforms to

$$u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \tag{3}$$

$$u'(0) = 0, \tag{4}$$

$$u(1) = \beta, \tag{5}$$

where

$$p(x) = \begin{cases} 0, & x = 0, \\ \dfrac{k}{x}, & x \neq 0, \end{cases}$$

$$q(x) = \begin{cases} \dfrac{b(0)}{k+1}, & x = 0, \\ b(x), & x \neq 0 \end{cases}$$

and

$$r(x) = \begin{cases} \dfrac{c(0)}{k+1}, & x = 0, \\ c(x), & x \neq 0. \end{cases}$$

We now partition the interval $[0, 1]$ using equally spaced knots $x_i = ih$, $i = 0, 1, 2, \ldots, n$, where $h = \dfrac{1}{n}$ and $n$ is an arbitrary positive integer. Let $u_i$ be an approximation to $u(x_i)$, obtained by the segment $S(x)$ of the spline function passing through the points $(x_i, u_i)$ and $(x_{i+1}, u_{i+1})$, where $S(x)$ is as defined in Taiwo and Ogunlaran [13].

Following Taiwo and Ogunlaran [13], one sided limits of derivative of $S(x)$ are

$$S'(x_i^-) = \frac{1}{h}(u_i - u_{i-1}) + h\beta M_i + h\alpha M_{i-1}, \quad i = 1, 2, \ldots, n \tag{6}$$

and

$$S'(x_i^+) = \frac{1}{h}(u_{i+1} - u_i) + h\alpha M_{i+1} + h\beta M_i, \quad i = 0, 1, 2, ..., n-1. \tag{7}$$

Substituting (6) and (7) in the differential equation (3) and rearranging, we obtain

$$(hq_i - p_i) + p_i u_{i+1} + (h - \beta h^2 p_i) M_i - \alpha h^2 p_i M_{i+1} = hr_i,$$
$$i = 0, 1, ..., n-1 \tag{8}$$

and

$$-p_i u_{i-1} + (hq_i + p_i) u_i + \alpha h^2 p_i M_{i-1} + (h + \beta h^2 p_i) M_i = hr_i,$$
$$i = 1, 2, ..., n. \tag{9}$$

From the boundary conditions (2) and (3), we obtain, respectively,

$$\frac{1}{h}u_1 - \frac{1}{h}u_0 - \beta h M_0 - \alpha h M_1 = 0 \tag{10}$$

and

$$u_n = B. \tag{11}$$

Equations (8)-(11) constitute $2(n+1)$ equations and therefore can be solved for the unknowns $u_i,\ M_i\ (i = 0, 1, ..., n)$.

## 3. Truncation Error

The local truncation error of (8) is given by

$$T_i = [hq(x_i) - p(x_i)]u(x_i) + p(x_i)u(x_{i+1}) + [h - \beta h^2 p(x_i)]u''(x_i)$$
$$- \alpha h^2 p(x_i) u''(x_{i+1}) - hr(x_i). \tag{12}$$

Expanding in Taylor's series about $x_i$, we obtain

$$T_i = h^2 p(x_i)\left[\frac{1}{2} - (\alpha + \beta)\right]u''(x_i) + h^3 p(x_i)\left[\frac{1}{6} - \alpha\right]u'''(x_i)$$

$$+ h^4 p(x_i)\left[\frac{1}{24} - \frac{\alpha}{2}\right]u^{iv}(x_i)$$

$$+ h^5 p(x_i)\left[\frac{1}{120} - \frac{\alpha}{6}\right]u^{v}(x_i) + O(h^6), \quad i = 0, 1, \ldots, n-1. \tag{13}$$

The same result is obtained by using (9). Thus, the schemes (8)-(11) produce a family of second-order methods for arbitrary choice of $\alpha$ and $\beta$ provided that $\alpha + \beta = \frac{1}{2}$. We consider the cases $\alpha = 1/4, 1/6, 1/8, 1/10, 1/12, 1/14$ and $1/16$.

## 4. Numerical Experiments

In this section, we have applied the new methods presented in this paper to solve three singular boundary value problems, both homogeneous and non-homogeneous, in order to demonstrate the applicability and the comparative performance of the methods.

**Problem 1.** Consider the linear singular boundary value problem

$$-u''(x) - \frac{2}{x}u'(x) + (1 - x^2)u(x) = x^4 - 2x^2 + 7$$

together with the boundary conditions

$$u'(0) = 0, \quad u(1) = 0.$$

The exact solution is

$$u(x) = 1 - x^2.$$

Each of the methods approximates the solution to this problem exactly. The solutions for $h = 1/4$ are presented in Table 1.

**Table 1.** Solution for Problem 1

| $x$ | Exact solution | Approximate solution |
|---|---|---|
| 0.0 | 1.0000 | 1.0000 |
| 0.25 | 0.9375 | 0.9375 |
| 0.50 | 0.7500 | 0.7500 |
| 0.75 | 0.4375 | 0.4375 |
| 1.00 | 0.0000 | 0.0000 |

**Table 2.** Observed maximum absolute errors for Problem 2

| $h\backslash$ Methods | $\alpha = 1/4$ | $\alpha = 1/6$ | $\alpha = 1/8$ | $\alpha = 1/10$ | $\alpha = 1/12$ | $\alpha = 1/14$ | $\alpha = 1/16$ |
|---|---|---|---|---|---|---|---|
| 1/8 | 5.4274E-4 | 1.7963E-4 | 5.1459E-6 | 1.1148E-4 | 1.8434E-4 | 2.3641E-4 | 2.7547E-4 |
| 1/16 | 1.3516E-4 | 4.4161E-5 | 1.5481E-6 | 2.8673E-5 | 4.6887E-5 | 5.9898E-5 | 6.9657E-5 |
| 1/32 | 3.3746E-5 | 1.0984E-5 | 4.1090E-7 | 7.2285E-6 | 1.1782E-5 | 1.5034E-5 | 1.7474E-5 |
| 1/64 | 8.4331E-6 | 7.7417E-6 | 1.0480E-7 | 1.8115E-6 | 2.9498E-6 | 3.7629E-6 | 4.3728E-6 |
| 1/128 | 2.1080E-6 | 6.8513E-7 | 2.6367E-8 | 4.5319E-7 | 7.3777E-7 | 9.4105E-7 | 1.0935E-6 |
| 1/256 | 5.2699E-7 | 1.7126E-7 | 6.6048E-9 | 1.1332E-7 | 1.8447E-7 | 2.3528E-7 | 2.7340E-7 |

**Problem 2.** Consider the Bessel's equation

$$u''(x) + \frac{1}{x}u'(x) + u(x) = 0$$

with boundary conditions

$$u'(0) = 0 \quad u(1) = 1$$

for which the exact solution is

$$u(x) = \frac{J_0(x)}{J_0(1)}.$$

The observed maximum absolute errors for our methods are summarized in Table 2. The methods are compared with the methods in Kanth and Reddy [1, 3] and, Caglar and Caglar [2] in Tables 3 and 4 from which it is seen that our methods with $\alpha = 1/8$ and $\alpha = 1/10$ perform better than the two methods. Figures 1 and 2 show the comparison of the exact and approximate solutions for Problem 2.

**Table 3.** Solution for Problem 2

| $x$ | Exact solution | $h = \frac{1}{10}$ | $h = \frac{1}{20}$ | |
|---|---|---|---|---|
| | | $\alpha = 1/8$ | [1] | $\alpha = 1/8$ |
| 0.0 | 1.306852 | 1.306854 | 1.306843 | 1.306852 |
| 0.1 | 1.303587 | 1.303590 | 1.303578 | 1.303588 |
| 0.2 | 1.293816 | 1.293819 | 1.293808 | 1.293817 |
| 0.3 | 1.277613 | 1.277616 | 1.277604 | 1.277614 |
| 0.4 | 1.255098 | 1.255102 | 1.255090 | 1.255099 |
| 0.5 | 1.226441 | 1.226444 | 1.226434 | 1.226442 |
| 0.6 | 1.191855 | 1.191858 | 1.191849 | 1.191856 |
| 0.7 | 1.151599 | 1.151602 | 1.151594 | 1.151600 |
| 0.8 | 1.105972 | 1.105974 | 1.105969 | 1.105973 |
| 0.9 | 1.055314 | 1.055315 | 1.055313 | 1.055314 |
| 1.0 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |

**Table 4.** Solution for Problem 2

| | Exact | $h = \frac{1}{20}$ | | | $h = \frac{1}{40}$ | | |
|---|---|---|---|---|---|---|---|
| $x$ | solution | [3] | [2] | $\alpha = 1/10$ | [3] | [2] | $\alpha = 1/10$ |
| 0.0 | 1.306852 | 1.306826 | 1.303823 | 1.306870 | 1.306840 | 1.306844 | 1.306856 |
| 0.1 | 1.303587 | 1.303562 | 1.303558 | 1.303605 | 1.303575 | 1.303579 | 1.303591 |
| 0.2 | 1.293816 | 1.293792 | 1.293789 | 1.293834 | 1.293804 | 1.293809 | 1.293820 |
| 0.3 | 1.277613 | 1.277589 | 1.277587 | 1.277629 | 1.277601 | 1.277606 | 1.277617 |
| 0.4 | 1.255098 | 1.255077 | 1.255075 | 1.255114 | 1.255088 | 1.255092 | 1.255102 |
| 0.5 | 1.226441 | 1.226422 | 1.226420 | 1.226455 | 1.226432 | 1.226435 | 1.226444 |
| 0.6 | 1.191855 | 1.191839 | 1.191838 | 1.191867 | 1.191848 | 1.191850 | 1.191858 |
| 0.7 | 1.151599 | 1.151586 | 1.151585 | 1.151608 | 1.151594 | 1.151595 | 1.151601 |
| 0.8 | 1.105972 | 1.105963 | 1.105963 | 1.105979 | 1.105969 | 1.105969 | 1.105974 |
| 0.9 | 1.055314 | 1.055309 | 1.055309 | 1.055317 | 1.055312 | 1.055312 | 1.055315 |
| 1.0 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |

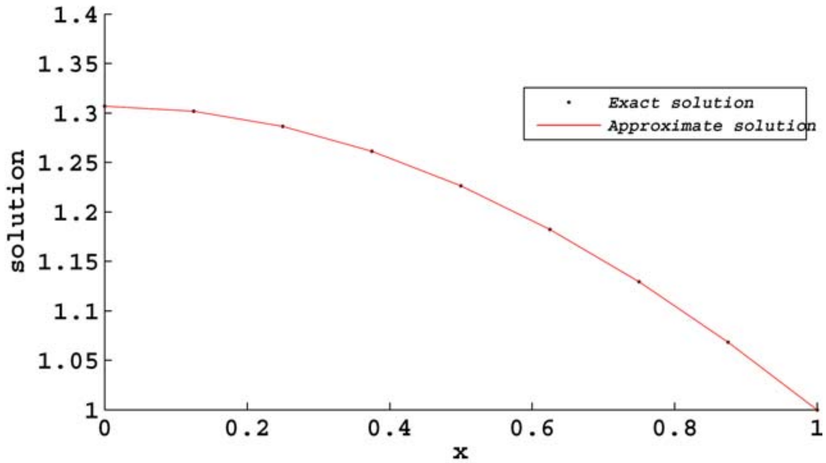


**Figure 1.** Comparison of solutions for Problem 2 with $\alpha = 1/8$ and $h = \frac{1}{8}$.

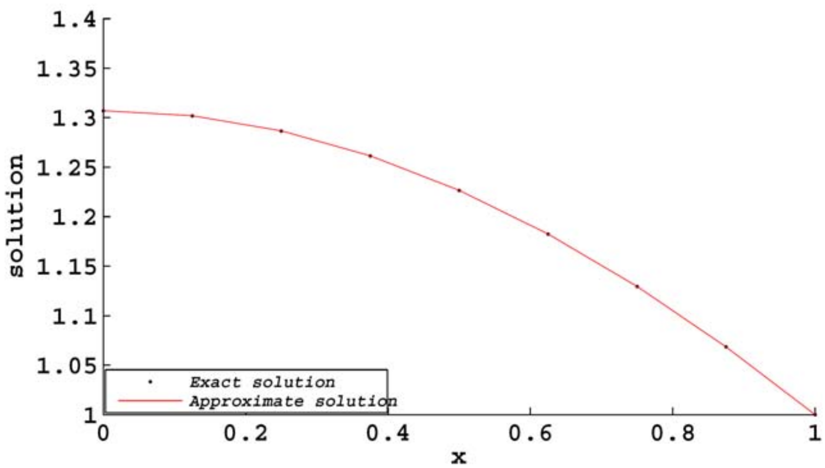


**Figure 2.** Comparison of solutions for Problem 2 with $\alpha = 1/8$ and $h = \frac{1}{8}$.

**Problem 3.** Consider

$$u''(x) + \frac{1}{x}u'(x) = \left(\frac{8}{8 - x^2}\right)^2,$$

$$u'(0) = 0, \quad u(1) = 0.$$

The exact solution is

$$u(x) = 2\log\frac{7}{8 - x^2}.$$

The observed maximum absolute errors for Problem 3 by the new methods are tabulated in Table 5. Table 6 shows that our method with $\alpha = 1/8$ performs better than Kanth and Reddy [1]. Similarly, Table 7 shows that our new methods with $\alpha = 1/10$ and $\alpha = 1/12$ produce better accurate results compared with Kanth and Reddy [3] and Caglar and Caglar [2]. Figure 3 shows the comparison of the exact and approximate solutions for this problem.

**Table 5.** Observed maximum absolute errors for Problem 3

| $h$ Methods | $\alpha = 1/4$ | $\alpha = 1/6$ | $\alpha = 1/8$ | $\alpha = 1/10$ | $\alpha = 1/12$ | $\alpha = 1/14$ | $\alpha = 1/16$ |
|---|---|---|---|---|---|---|---|
| 1/8 | 4.7804E-4 | 1.7304E-4 | 2.0539E-5 | 7.1223E-5 | 1.3196E-4 | 1.7553E-4 | 2.0821E-4 |
| 1/16 | 1.1893E-4 | 4.2566E-5 | 4.3859E-6 | 1.8538E-5 | 3.3794E-5 | 4.4703E-5 | 5.2885E-5 |
| 1/32 | 2.9688E-5 | 1.0590E-5 | 1.0418E-6 | 4.6883E-6 | 8.5067E-6 | 1.1235E-5 | 1.3281E-5 |
| 1/64 | 7.4186E-6 | 2.6439E-6 | 2.5655E-7 | 1.1759E-6 | 2.1308E-6 | 2.8129E-6 | 3.3245E-6 |
| 1/128 | 1.8544E-6 | 6.6072E-7 | 6.3864E-8 | 2.9425E-7 | 5.3299E-7 | 7.0352E-7 | 8.3142E-7 |
| 1/256 | 4.6359E-7 | 1.6516E-7 | 1.5947E-8 | 7.3582E-8 | 1.3327E-7 | 1.7590E-7 | 2.0787E-7 |

**Table 6.** Solution for Problem 3

| $x$ | Exact solution | $h = \frac{1}{20}$ | |
|---|---|---|---|
| | | $\alpha = 1/8$ | [1] |
| 0.0 | –0.267063 | –0.267066 | –0.267067 |
| 0.1 | –0.264561 | –0.264564 | –0.264565 |
| 0.2 | –0.257038 | –0.257040 | –0.257142 |
| 0.3 | –0.244435 | –0.244438 | –0.244439 |
| 0.4 | –0.226657 | –0.226660 | –0.226661 |
| 0.5 | –0.203565 | –0.203568 | –0.203569 |

| | | | |
|---|---|---|---|
| 0.6 | –0.174975 | –0.174977 | –0.174978 |
| 0.7 | –0.140651 | –0.140653 | –0.140653 |
| 0.8 | –0.100300 | –0.100301 | –0.100301 |
| 0.9 | –0.053562 | –0.053563 | –0.053563 |
| 1.0 | 0 | 0 | 0 |

**Table 7.** Solution for Problem 3

| | Exact | $h = \frac{1}{20}$ | | | | $h = \frac{1}{40}$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | solution | $\alpha = 1/10$ | $\alpha = 1/12$ | [3] | [2] | $\alpha = 1/10$ | $\alpha = 1/12$ | [3] | [2] |
| 0.0 | -0.267063 | -0.267051 | -0.267041 | -0.267089 | -0.267089 | -0.267060 | -0.267057 | -0.267068 | -0.267069 |
| 0.1 | -0.264561 | -0.264549 | -0.264540 | -0.264588 | -0.264588 | -0.264558 | -0.264556 | -0.264567 | -0.264567 |
| 0.2 | -0.257038 | -0.257026 | -0.257017 | -0.257063 | -0.257064 | -0.257035 | -0.257032 | -0.257043 | -0.257044 |
| 0.3 | -0.244435 | -0.244424 | -0.244415 | -0.244460 | -0.244460 | -0.244433 | -0.244430 | -0.244440 | -0.244441 |
| 0.4 | -0.226657 | -0.226647 | -0.226639 | -0.226681 | -0.226681 | -0.226653 | -0.226653 | -0.226662 | -0.226663 |
| 0.5 | -0.203565 | -0.203556 | -0.203549 | -0.203587 | -0.203587 | -0.203563 | -0.203561 | -0.203570 | -0.203570 |
| 0.6 | -0.174975 | -0.174967 | -0.174960 | -0.174994 | -0.174994 | -0.174973 | -0.174971 | -0.174979 | -0.174979 |
| 0.7 | -0.140651 | -0.140644 | -0.140639 | -0.740666 | -0.140666 | -0.140649 | -0.140648 | -0.140654 | -0.140654 |
| 0.8 | -0.100300 | -0.100295 | -0.100291 | -0.100311 | -0.100311 | -0.100298 | -0.100297 | -0.100302 | -0.100302 |
| 0.9 | -0.053562 | -0.053560 | -0.053558 | -0.053568 | -0.053568 | -0.053561 | -0.053561 | -0.053564 | -0.053563 |
| 1.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

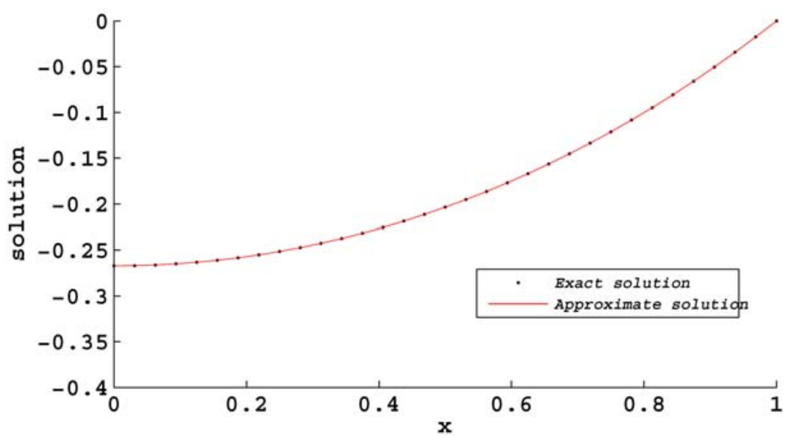


**Figure 3.** Comparison of solutions for Problem 3 with $\alpha = 1/12$ and $h = \frac{1}{32}$.

## 5. Conclusion

Numerical methods based on a non-polynomial spline have been developed for the solution of both homogeneous and nonhomogeneous linear second-order singular boundary value problems. The methods are second-order accurate as confirmed by the results. The methods are computationally efficient and the schemes can be easily implemented on a computer. The methods have also been shown to be more accurate than some existing methods. The accuracy of the methods can still be improved on by applying extrapolation technique to the methods.

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