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A Computational Method for System of Linear Fredholm Integral Equations

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Abstract

This paper focuses on developing a numerical method based on a cubic spline approach for the solution of system of linear Fredholm equations of the second kind. This method produces a system of algebraic equations. The efficiency and accuracy of the method are demonstrated by a numerical example and the mathematical software Matlab R2010a was used to carry out the necessary computations.

Keywords: System of linear Fredholm integral equations, natural cubic spline

1. Introduction

Many mathematical formulations of physical phenomena involve integral equations. These equations arise frequently in applied areas including engineering, mechanics, biology, chemistry, physics, potential theory, electrostatics and economics (Ren et al. 1999; Rashed 2004; Atkinson 1997; Wazwaz 2011; Delves & Mohamed 1985; Linz 1985). Integral equations also occur as reformulations of differential equations. However, most integral equations that arise in real life situations are difficult to solve analytically; therefore a numerical method is required.

In recent years, a numerous numerical methods have been developed for solving various types of integral equations such as rationalized Haar function method (Ordokhani & Razzaghi 2008), Haar wavelet method (Mishra et al. 2012; Lepik & Tamme 2004), Adomain method (El-Kalla 2008), Sinc Collocation method (Rashidinia & Zerebnia 2005; Maleknejad & Nedaiasi 2011), A variation of Nystroms method (Lardy 1981), Petrov-Galerkin method (Kaneko et al. 2003).

We consider the following system of linear Fredholm equations of the second kind:

$$\left.\begin{aligned} x(t)-\int_a^b k_1(t,s)\big(x(s)+y(s)\big)\,ds &= f(t)\\ y(t)-\int_a^b k_2(t,s)\big(x(s)+y(s)\big)\,ds &= g(t)\end{aligned}\right\} \qquad (1)$$

where the functions $f(t),\ g(t),\ k_1(s,t)$ and $k_2(s,t)$ are known , and $x(t)$ and $y(t)$ are the unknowns to be determined.

2. The Solution Method

In this section, a cubic spline method is applied to solve (1). In the first place, to solve (1) in the interval [a,b], we partition the range into smaller intervals of uniform width h such that $s_i = s_0 + ih,\ i = 0,1,\ \cdots, n,\ s_0 = a,\ s_n = b$ and $nh = b - a$.

Setting $t = t_{j,}$ (1) may be written as follows

$$\left.\begin{aligned} x(t_j) - \sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} k_1(t_j,s)\big(x(s)+y(s)\big)\,ds = f(t_j) \\ y(t_j) - \sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} k_2(t_j,s)\big(x(s)+y(s)\big)\,ds = g(t_j) \end{aligned}\right\} \qquad (2)$$

Following Taiwo & Ogunlaran (2008), we approximate the integrals terms of 2(a) by the cubic spline to obtain

$$x(t) - \sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} k_1(t_j,s)\Bigg\{\Bigg[\frac{1}{6h}(s_{i+1}-s)^3 C_i + \frac{1}{6h}(s-s_i)^3 C_{i+1} + \left(\frac{x_i}{h} - \frac{hC_i}{6}\right)(s_{i+1}-s)$$

$$\left(\frac{x_{i+1}}{h} - \frac{hC_{i+1}}{6}\right)(t-t_i)\Bigg] + \Bigg[\frac{1}{6h}(s_{i+1}-s)^3 M_i + \frac{1}{6h}(s-s_i)^3 M_{i+1} + \left(\frac{y_i}{h} - \frac{h}{6}M_i\right)(s_{i+1}-s)$$

$$+\left(\frac{y_{i+1}}{h} - \frac{hM_{i+1}}{6}\right)(s-s_i)\Bigg]\Bigg\} ds = f(t_j), \qquad j = 0,1,\ \cdots, n.$$

Now substituting $s = s_i + ph$ and simplifying, we obtain

$$x(t_j) - \sum_{i=0}^{n-1}\int_0^1 k_1(t_j, s_i + ph)\Bigg[h(1-p)x_i + hpx_{i+1} + h(1-p)y_i + hpy_{i+1} + \frac{h^3}{6}(-2p+3p^2-p^3)(C_i+M_i)$$

$$\frac{h^3}{6}(p^3-p)\big(C_{i+1}+M_{i+1}\big)\Bigg]dp = f(t_j), \qquad j = 0,1,\cdots,n \qquad (3a)$$

Similarly from (2b) we obtain

$$y(t_j)-\sum_{i=0}^{n-1}\int_0^1 k_2(t_j,s_i+ph)\Big[h(1-p)x_i+hpx_{i+1}+h(1-p)y_i+hpy_{i+1}+\frac{h^3}{6}(-2p+3p^2-p^3)(C_i+M_i)$$
$$\frac{h^3}{6}(p^3-p)\left(C_{i+1}+M_{i+1}\right)\Big]dp=g(t_j),\qquad j=0,1,\cdots,n \tag{3b}$$

Furthermore, following Taiwo & Ogunlaran (2008) we have the following consistency relations:

$$C_{i-1}+4C_i+C_{i+1}=\frac{6}{h^2}\left(x_{i-1}-2x_i+x_{i+1}\right),\quad i=1,2,\cdots,n\text{-}1. \tag{4}$$

and

$$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}\left(y_{i-1}-2y_i+y_{i+1}\right),\quad i=1,2,\cdots,n\text{-}1. \tag{5}$$

To have a complete system, we impose the end conditions

$$C_0=C_n=0 \quad\text{and}\quad M_0=M_n=0 \tag{6}$$

These boundary conditions are referred to as free/natural boundary conditions. When free boundary conditions occur the spline is called a natural spline (Burden & Faires 1993).

Equations (3) – (6) give $4(n+1)$ equations which is solved to determine the $4(n+1)$ unknowns $x_j,\ y_j, C_j,\ M_j,\ j=0,1,\ \cdots,\ n.$

3. Illustrative Example

In this section we applied the method presented in this paper to solve an example. The example chosen has exact solutions and has earlier been solved by Vahidi & Mokhtari (2008). The results in terms of the observed errors in absolute values are presented in Table 1 and the exact and numerical solutions are further illustrated and compared in Figure 1.

Consider the following system of linear Fredholm integral equations

$$x(t)=\frac{t}{18}+\frac{17}{36}+\int_0^1\frac{s+t}{3}\left(x(s)+y(s)\right)ds$$
$$\mathrm{y}(t)=t^2-\frac{19t}{12}+1+\int_0^1 st\left(x(s)+y(s)\right)ds$$

The exact solutions to this problem are $x(t)=t+1$ and $y(t)=t^2+1$

4. Conclusion

Natural Cubic spline has been applied to solve system of linear Fredholm integral equations of second kind. The numerical results show that the method is applicable and efficient. Also, as expected, the accuracy of the method improves with increasing number of subintervals n. This new method performs better in terms of accuracy compared to Adomain decomposition method and it can easily be extended to solve an n dimensional system of first order Fredholm integral equations.

Table 1: Absolute Errors for the Problem

t	$x(t)$				$y(t)$			
	Vahidi & Mokhtari	Our method			Vahidi & Mokhtari	Our method		
		$n=5$	$n=10$	$n=20$		$n=5$	$n=10$	$n=20$
0	1.15E-2	4.19E-4	5.22E-5	6.53E-6	0	0	0	0
0.1	1.33E-2	-	6.13E-5	7.66E-6	3.45E-3	-	1.57E-5	1.96E-6
0.2	1.52E-2	5.65E-4	7.04E-5	8.80E-6	6.90E-3	2.51E-4	3.13E-5	3.92E-6
0.3	1.71E-2	-	7.95E-4	9.93E-6	1.03E-2	-	4.70E-5	5.88E-6
0.4	1.89E-2	7.10E-4	8.85E-5	1.11E-5	1.38E-2	5.03E-4	6.27E-5	7.84E-6
0.5	2.08E-2	-	9.76E-5	1.22E-5	1.72E-2	-	7.84E-5	9.79E-6
0.6	2.26E-2	8.56E-4	1.07E-4	1.33E-5	2.07E-2	7.54E-4	9.40E-5	1.18E-5
0.7	2.45E-2	-	1.16E-4	1.45E-5	2.41E-2	-	1.10E-4	1.37E-5
0.8	2.64E-2	1.00E-3	1.25E-4	1.56E-5	2.76E-2	1.01E-3	1.25E-4	1.57E-5
0.9	2.82E-2	-	1.34E-4	1.67E-5	3.10E-2	-	1.41E-4	1.76E-5
1.0	3.02E-1	1.15E-3	1.43E-4	1.79E-5	3.45E-2	1.26E-3	1.57E-4	1.96E-5

Table 1 shows the comparison of absolute errors in solutions by using our method with various values of n and the Adomain decomposition method (Vahidi & Mokhtari 2008) for the same problem at the eleventh iteration. It is observed from the Table that the results by the spline method are better compared to Adomain decomposition method, even with $n=5$, for both $x(t)$ and $y(t)$.

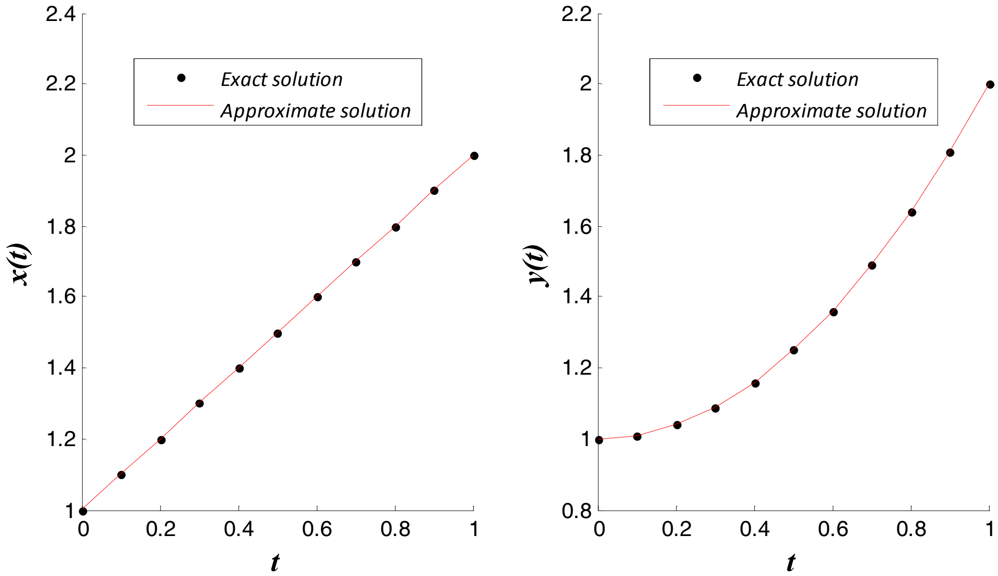

Figure 1: Plots of the Exact and Approximate Solutions for n=10

Figure 1 shows the graphs of the exact solutions and approximate solutions obtained by using the new method with $n=10$ for both $x(t)$ and $y(t)$. From the Figure, approximate solutions compare favourably with the exact solutions which confirm the accuracy of the new method.

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