J. Math. Computer Sci., 29 (2023), 60–72

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Approximate solutions of linear time-fractional differential equations

Razaq Adekola Oderinu[a], Johnson Adekunle Owolabi[b,*], Musilimu Taiwo[c]

[a]Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, P.M.B 4000, Ogbomoso, Oyo Sate, Nigeria.

[b]Department of Mathematics, Bowen University, Iwo, P.M.B 284, Osun State, Nigeria.

[c]Department of Mathematics, Osun State College of Education, PMB 207, Ila-Orangun, Osun State, Nigeria.

Abstract

In this research work, the numerical scheme for obtaining the linear time-fractional differential equations was considered and the nature of these time-fractional differential equations are in sense of Caputo. A theorem was proved to show the Kamal transform of nth order Caputo derivatives. Finally, three problems were considered regarding the linear time-fractional differential equations which presented that the convergence of the scheme provided in the research are of high accuracy for solving and linear fractional differential equations.

Keywords: Kamal transform, adomian polynomial, linear time-fractional differential equations

2020 MSC: 26A33, 35R11.

1. Introduction

The interest of researchers in handling fractional differential equations has greatly increased due to their applications in the field of science and engineering. Some of these applications are found in Physics, Chemistry, Signal and Image processing, Economics, Biology and so on [4, 7, 18, 19, 27, 29, 30]. Recently, it was investigated that fractional differential equations are used to model real world problems in accurate way as when compared with classical order and these has motivated researchers to be looking for a way to handle these fractional order differential equations, this is because there is no precise method that will yield an exact solution for fractional differential equation, it is only approximate solutions that would be derived in solving it [2, 20, 23, 24, 36].

Several methods has been used to obtain the approximate analytical solution of these so called fractional differential equation. Some of the methods used include: Shehu decomposition and variational iteration [6], Laplace modified Adomian decomposition method [3], Laplace Adomian decoposition method [2, 4, 16, 17, 29, 30], Adomian decomposition method [10, 11, 20, 22–24, 27, 28, 33, 34], homotopy analysis method [13], Laplace decomposition method [12, 21], variational iteration method [35], homotopy perturbation method [5], iterative Laplace transform method [32], and multi-stem Laplace decomposition method [36].

*Corresponding author

Email addresses: raoderinu@lautech.edu.ng (Razaq Adekola Oderinu), johnson.owolabi@bowen.edu.ng (Johnson Adekunle Owolabi), taiwomuslim@gmail.com (Musilimu Taiwo)

doi: 10.22436/jmcs.029.01.06

Received: 2021-12-23 *Revised:* 2022-02-01 *Accepted:* 2022-03-16

2. Definitions

The Caputo definition has the advantage of dealing properly with the initial value problems, in which the initial conditions are given in terms of the field variables and their integral order, which is the case in most physical process.

Definition 2.1. The Riemann-Liouville fractional derivative of order α is defined as [10, 24]

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t).$$

where $\alpha > 0$, $[0, t]$ is the interval, $\Gamma(\cdot)$ denotes the Gamma function and $*$ is the convolution operator.

Definition 2.2. The Caputo fractional derivative of order α is defined as [20, 27]

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}ds,$$

where $n-1 < \alpha \leqslant n$, and n is an integer.

3. The concept of Kamal transform

Kamal transform is defined for the function of exponential order [1, 25, 26]. Consider the functions in the set S defined by

$$S = \left\{f(t) : k_1, k_2 > 0, |f(t)| < Me^{|t|/k_j}, t \in (-1)^j \times [0, \infty)\right\},$$

where M is a constant which must be a finite number, k_1, k_2 are also constants which can be finite or infinite. Kamal transform is denoted by the operator $K(\cdot)$ which is defined as:

$$K\left[f(t)\right] = \int_0^{\infty} f(t)e^{-t/\nu}dt = G(\nu), \qquad t \geqslant 0, \quad k_1 \leqslant \nu \leqslant k_2.$$

Theorem 3.1. *Let* $G(\nu)$ *be a Kamal transform of* $f(t)$*, where* $K[f(t)] = G(\nu)$*, then*

1. $K[f'(t)] = \frac{1}{\nu}G(\nu) - f(0)$;
2. $K[f''(t)] = \frac{1}{\nu^2}G(\nu) - \frac{1}{\nu}f(0) - f'(0)$;
3. $K[f^n(t)] = \frac{1}{\nu^n}G(\nu) - \sum_{k=0}^{n-1}\nu^{k-n+1}f^k(0)$.

The proof of Theorem 3.1 is in [25].

Theorem 3.2. *Given the* n^{th} *order Caputo fractional derivative, then the Kamal transform is given by*

$$K\left[{}^cD_x^{\alpha}f(x)\right] = \nu^{-\alpha}G(\nu) - \sum_{k=0}^{n-1}\nu^{k-\alpha+1}f^k(0) \quad n-1 < \alpha < n, \ \ n \in \mathbb{N}.$$

Proof. Recalled the fractional derivative $f(x)$ in the Caputo sense as given by [14], then,

$$K\left[{}^cD_x^{\alpha}f(x)\right] = K\left[I_x^{n-\alpha}f^n(x)\right].$$

Here, we need to show that

$$K\left[I_x^{n-\alpha}f(x)\right] = \nu^{\alpha}F(\nu),$$

so that

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad x > 0, \quad \alpha > 0,$$

and by convolution theorem,

$$f * g(x) = \int_0^x f(x-t) * g(t)dt,$$

and so,

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt = \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} * f(x)\right). \tag{3.1}$$

Then, equation (3.1) becomes

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} * f(x)\right). \tag{3.2}$$

Taking the Kamal transform of equation (3.2) gives

$$K\left[I_x^\alpha f(x)\right] = \frac{1}{\Gamma(\alpha)} K\left[x^{\alpha-1}\right] * F(v) = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha)\, v^\alpha * F(v).$$

Therefore,

$$K\left[I_x^\alpha f(x)\right] = v^\alpha * F(v).$$

Now, if $f^n(x) = g(x)$, then we have

$$K\left[I_x^{n-\alpha} f^n(x)\right] = K\left[I_x^{n-\alpha} g(x)\right] = v^{n-\alpha} G(v). \tag{3.3}$$

Applying Theorem 3.1 on equation (3.3) gives

$$K\left[I_x^{n-\alpha} f^n(x)\right] = v^{n-\alpha} \left(\frac{1}{v^n} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} f^k(0)\right).$$

Therefore the Kamal transform of the Caputo fractional derivative is defined as:

$$K\left[{}^c D_x^\alpha f(x)\right] = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-\alpha+1} f^k(0).$$

□

4. The Kamal decomposition transform method on fractional differential equations

Consider the general fractional differential equation given as ([4])

$$\begin{aligned} D_x^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t),\ t > 0,\ 0 < \alpha \leqslant 1, \\ u^k(x,0) = r(x), \forall\, x \in \mathbb{R},\ k = 1,2,3,\ldots, \end{aligned} \tag{4.1}$$

where $D_x^\alpha u(x,t)$ is the Caputo fractional derivative of order α, $Ru(x,t)$ is the linear operator with fractional derivative order less than α, $Nu(x,t)$ is a nonlinear operator which also might include other derivative of order less than α, and $g(x,t)$ is known as the analytical function.

Taking the Kamal transform of equation (4.1) gives

$$K\left[D_x^{\alpha}u(x,t)\right]+K\left[Ru(x,t)+Nu(x,t)\right]=K\left[g(x,t)\right]. \tag{4.2}$$

Simplifying equation (4.2) gives

$$v^{-\alpha}K\left[u(x,t)\right]-\sum_{k=0}^{n=1}v^{k-\alpha+1}u^{k}(0)+K\left[Ru(x,t)+Nu(x,t)\right]=K\left[g(x,t)\right]. \tag{4.3}$$

Thus, equation (4.3) becomes

$$K\left[u(x,t)\right]-v^{\alpha}\sum_{k=0}^{n=1}v^{k-\alpha+1}u^{k}(0)+v^{\alpha}K\left[Ru(x,t)+Nu(x,t)\right]=v^{\alpha}K\left[g(x,t)\right]. \tag{4.4}$$

Introducing the inverse Kamal transform on equation (4.4) gives

$$u(x,t)=K^{-1}\left[v^{\alpha}\left\{\sum_{k=0}^{n=1}v^{k-\alpha+1}u^{k}(0)+K\left[g(x,t)\right]\right\}\right]+K^{-1}\left[v^{\alpha}K\left\{Ru(x,t)+Nu(x,t)\right\}\right]. \tag{4.5}$$

Given that

$$u(x,t)=h(x,t)+K^{-1}\left[v^{\alpha}K\left\{Ru(x,t)+Nu(x,t)\right\}\right],$$

where $h(x,t)$ denotes the expression that arises from the given initial condition and the source term after simplification. The solution will be in the form of infinite series given as

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t).$$

Decomposing the nonlinear term in equation (4.2) gives

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n,$$

where A_n is defined by the Adomian polynomial which can be calculated with the aid of formula [25, 26]. That is

$$A_n=\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0},\quad n=0,1,\ldots.$$

Then from equation (4.5),

$$u_0(x,t)=K^{-1}\left[v^{\alpha}\left\{\sum_{k=0}^{n=1}v^{k-\alpha+1}u^{k}(0)+K\left[g(x,t)\right]\right\}\right],$$

and the recursive relation is given as

$$u_{n+1}=K^{-1}\left[v^{\alpha}K\left\{Ru(x,t)+A_n\right\}\right].$$

The analytical solution $u(x,t)$ can be approximated by a truncated series

$$u(x,t)=\lim_{N\to\infty}\sum_{n=0}^{N}u_n(x,t).$$

5. Applications

Problem 5.1. Consider the linear time fractional Klein-Gordon equation ([4])

$$D_x^\alpha - u_{xx} - u = 0, \qquad 1 < \alpha \leqslant 2, \tag{5.1}$$

with initial condition $u(x,0) = 0, u_t(x,0) = x$. Taking Kamal transform to both sides of equation (5.1) gives

$$K\left[D_x^\alpha\right] = K\left[u_{xx} - u\right]. \tag{5.2}$$

Equation (5.2) becomes

$$K\left[u(x,t)\right] = v^\alpha \sum_{k=0}^{1} v^{k-\alpha+1} u^k(x,0) - v^\alpha K\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right]. \tag{5.3}$$

Then, equation (5.3) becomes

$$K\left[u(x,t)\right] = v^\alpha \left(v^{-\alpha+1} u(x,0) + v^{2-\alpha} u'(x,0)\right) - v^\alpha K\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right].$$

Applying the initial conditions given, then we have

$$K\left[u(x,t)\right] = v^\alpha \left(v^{2-\alpha} x\right) - v^\alpha K\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right]. \tag{5.4}$$

Introducing the inverse Kamal transform on equation (5.4) and simplifying gives

$$u(x,t) = K^{-1}\left[v^2 x\right] - k^{-1}\left[v^\alpha K\left[\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\right]\right]. \tag{5.5}$$

From equation (5.5),

$$u_0(x,t) = K^{-1}\left[v^2 x\right] = tx.$$

Now the recursive relation is given as

$$u_{n+1} = -k^{-1}\left[v^\alpha K\left[\frac{\partial^2 u_n(x,t)}{\partial x^2} + u_n(x,t)\right]\right]. \tag{5.6}$$

If $n = 0$ in equation (5.6), then

$$u_1(x,t) = -k^{-1}\left[v^\alpha K\left[\frac{\partial^2 u_0(x,t)}{\partial x^2} + u_0(x,t)\right]\right]. \tag{5.7}$$

Simplifying equation (5.7) gives

$$u_1(x,t) = -k^{-1}\left[v^\alpha v^2 x\right] = v^\alpha t x = \frac{t^{\alpha+1} x}{\Gamma(\alpha+2)}.$$

If $n = 1$ in equation (5.6), then

$$u_2(x,t) = -k^{-1}\left[v^\alpha K\left[\frac{\partial^2 u_1(x,t)}{\partial x^2} + u_1(x,t)\right]\right]. \tag{5.8}$$

Simplifying equation (5.8) gives

$$u_2(x,t) = -k^{-1}\left[\frac{v^{\alpha}(\alpha+1)!xv^{\alpha+2}}{\Gamma(\alpha+2)}\right] = \frac{v^{\alpha}t^{\alpha+1}x}{\Gamma(\alpha+2)}.$$

Therefore

$$u_2(x,t) = \frac{t^{2\alpha+1}x}{\Gamma(2\alpha+2)}.$$

If $n=2$ in equation (5.6), then

$$u_3(x,t) = -k^{-1}\left[v^{\alpha}K\left[\frac{\partial^2 u_2(x,t)}{\partial x^2}+u_2(x,t)\right]\right]. \tag{5.9}$$

Simplifying equation (5.9) gives

$$u_3(x,t) = -k^{-1}\left[\frac{v^{\alpha}(2\alpha+1)!xv^{2\alpha+2}}{\Gamma(2\alpha+2)}\right] = -\frac{v^{\alpha}t^{2\alpha+1}x}{\Gamma(2\alpha+2)}.$$

Therefore

$$u_3(x,t) = -\frac{t^{3\alpha+1}x}{\Gamma(3\alpha+2)}.$$

Thus, the series solution becomes

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots.$$

Then

$$u(x,t) = xt - \frac{t^{\alpha+1}x}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}x}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}x}{\Gamma(3\alpha+2)} + \cdots. \tag{5.10}$$

If $\alpha=2$ in equation (5.10), then the close solution becomes

$$u(x,t) = x\sin t.$$

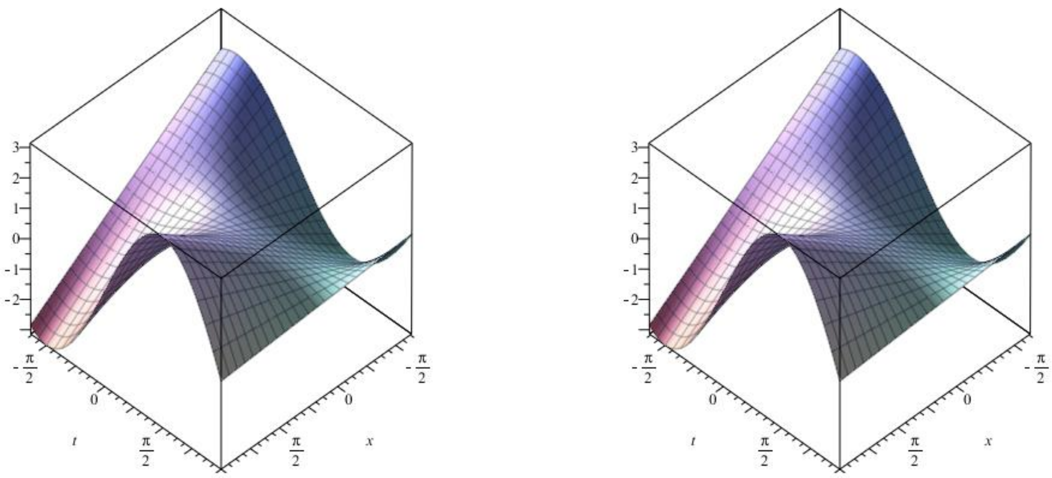
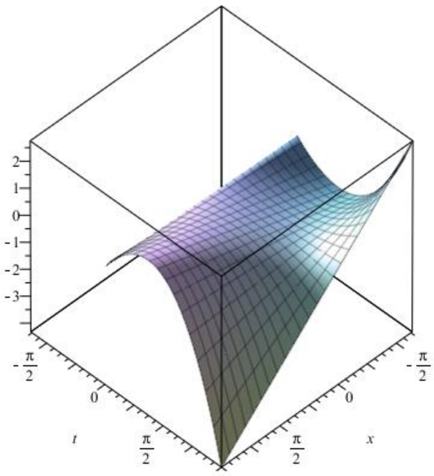


Figure 1: The left and right panel show the graphs of exact and Kamal transform method solution for $\alpha = 2$.



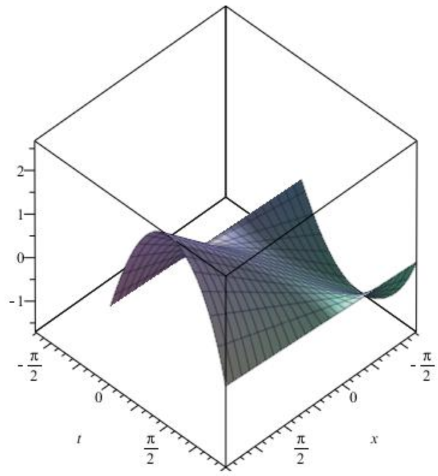


Figure 2: The left and right panel show the graphs of $\alpha = 0.5$ and $\alpha = 1.5$, respectively.

Problem 5.2. Consider the two-dimensional fractional wave equation ([15])

$$D_x^{\alpha} - \frac{y^2}{12}\frac{\partial^2 u}{\partial x^2} - \frac{x^2}{12}\frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < \alpha \leqslant 2, \tag{5.11}$$

with initial condition $u(x,y,0) = x^4, u_t(x,y,0) = y^4$. Taking Kamal transform to both sides of equation (5.11) gives

$$K\left[D_x^{\alpha}\right] = K\left[\frac{y^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{x^2}{12}\frac{\partial^2 u}{\partial y^2}\right]. \tag{5.12}$$

Equation (5.12) becomes

$$K\left[u(x,y,t)\right] = v^{\alpha}\sum_{k=0}^{1} v^{k-\alpha+1}u^{k}(x,y,0) + v^{\alpha}K\left[\frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2}\right]. \tag{5.13}$$

Then, equation (5.13) becomes

$$K\left[u(x,y,t)\right] = v^{\alpha}\left(v^{-\alpha+1}u(x,y,0) + v^{2-\alpha}u'(x,y,0)\right) + v^{\alpha}K\left[\frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2}\right].$$

Applying the initial conditions given, then we have

$$K\left[u(x,y,t)\right] = \left(vx^4 + v^2y^4x\right) + v^{\alpha}K\left[\frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2}\right]. \tag{5.14}$$

Introducing the inverse Kamal transform on equation (5.14) and simplifying gives

$$u(x,y,t) = K^{-1}\left[vx^4 + v^2y^4\right] + K^{-1}\left[v^{\alpha}K\left[\frac{x^2}{12}\frac{\partial^2 u}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u}{\partial y^2}\right]\right]. \tag{5.15}$$

From equation (5.15),

$$u_0(x,y,t) = K^{-1}\left[vx^4 + v^2y^4\right] = x^4 + ty^4.$$

Now the recursive relation is given as

$$u_{n+1} = K^{-1}\left[v^{\alpha}K\left[\frac{x^2}{12}\frac{\partial^2 u_n(x,y,t)}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_n(x,y,t)}{\partial y^2}\right]\right]. \tag{5.16}$$

If $n = 0$ in equation (5.16), then

$$u_1(x,y,t) = K^{-1}\left[v^{\alpha}K\left[\frac{x^2}{12}\frac{\partial^2 u_0(x,y,t)}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_0(x,y,t)}{\partial y^2}\right]\right]. \tag{5.17}$$

Simplifying equation (5.17) gives

$$u_1(x,y,t) = K^{-1}\left[v^{\alpha}K\left[x^4 + ty^4\right]\right] = K^{-1}\left[v^{\alpha}K\left[x^4\right]\right] + K^{-1}\left[v^{\alpha}K\left[ty^4\right]\right]. \tag{5.18}$$

Then equation (5.18) becomes

$$u_1(x,y,t) = v^{\alpha}x + v^{\alpha}y^4 t = \frac{x^4 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{y^4 t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

If $n = 1$ in equation (5.16), then

$$u_2(x,y,t) = K^{-1}\left[v^{\alpha}K\left[\frac{x^2}{12}\frac{\partial^2 u_1(x,y,t)}{\partial x^2} + \frac{y^2}{12}\frac{\partial^2 u_1(x,y,t)}{\partial y^2}\right]\right]. \tag{5.19}$$

Simplifying equation (5.19) gives

$$u_2(x,y,t) = K^{-1}\left[v^{\alpha}K\left[\frac{x^4 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{y^4 t^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right]. \tag{5.20}$$

Further simplification of equation (5.20) gives

$$u_2(x,y,t) = K^{-1}\left[\frac{v^{\alpha}\alpha! x^4 t^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{v^{\alpha}y^4(\alpha+1)! t^{\alpha+2}}{\Gamma(\alpha+2)}\right].$$

Then

$$u_2(x,y,t) = \frac{x^4 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{y^4 t^{2\alpha+1}}{\Gamma(2\alpha+2)}.$$

Thus, the series solution becomes

$$u(x,y,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.$$

Then

$$u(x,y,t) = x^4 + y^4 t + \frac{x^4 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{y^4 t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^4 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{y^4 t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots. \tag{5.21}$$

If $\alpha = 2$ in equation (5.21), then the close solution becomes

$$u(x,y,t) = x^4\cosh t + y^4\sinh t.$$

Problem 5.3. Consider the three-dimensional diffusion equation ([31])

$$D_x^{\alpha}u(x,y,z,t) = \frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}, \quad 0 < \alpha \leqslant 1, \tag{5.22}$$

with initial condition $u(x,y,z,0) = \sin x \sin y \sin z$. Taking Kamal transform to both sides of equation (5.22) gives

$$K\left[D_x^{\alpha}u(x,y,z,t)\right] = K\left[\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right]. \tag{5.23}$$

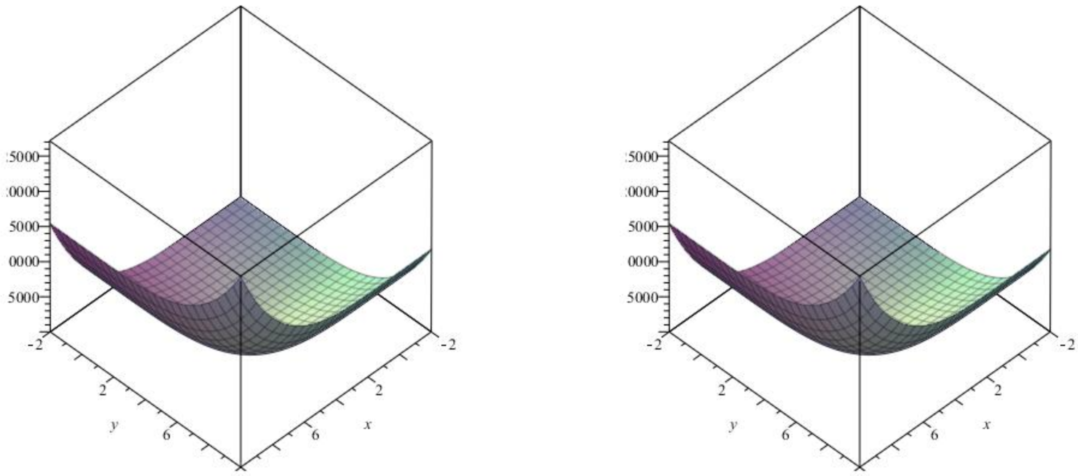


Figure 3: The left and right panel show the graphs of exact and Kamal transform method solution for $\alpha = 2$.

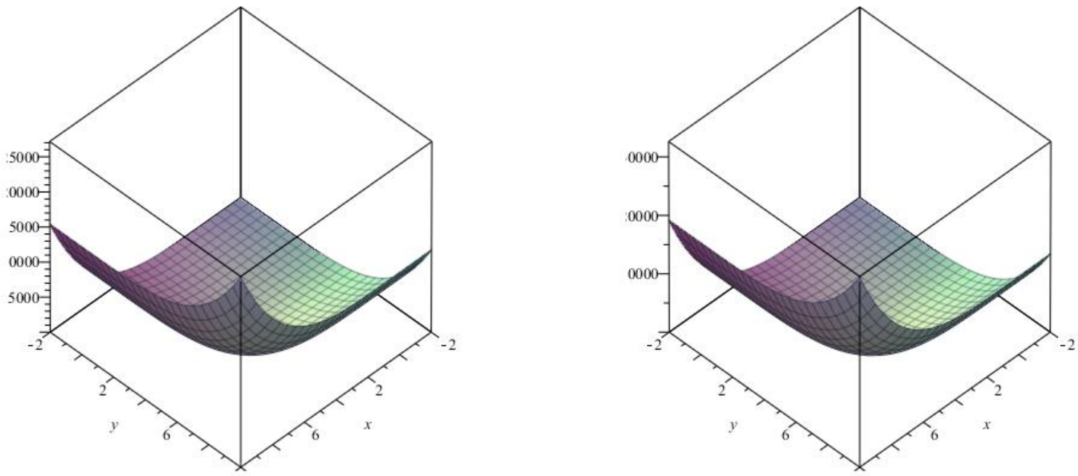


Figure 4: The left and right panel show the graphs of $\alpha = 0.75$ and $\alpha = 1.5$, respectively.

Equation (5.23) becomes

$$\begin{aligned} K\left[u(x,y,z,t)\right] =& \nu^{\alpha} \sum_{k=0}^{0} \nu^{k-\alpha+1} u^{k}(x,y,z,0) \\ &+ \nu^{\alpha} K\left[\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right]. \end{aligned} \tag{5.24}$$

Then, equation (5.24) becomes

$$K\left[u(x,y,z,t)\right] = \nu^{\alpha}\left(\nu^{-\alpha+1} u(x,y,z,0)\right) + \nu^{\alpha} K\left[\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right].$$

Applying the initial conditions given, then we have

$$K\left[u(x,y,z,t)\right] = \nu \sin x \sin y \sin z + \nu^{\alpha} K\left[\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right]. \tag{5.25}$$

Introducing the inverse Kamal transform on equation (5.25) and simplifying gives

$$u(x,y,z,t) = K^{-1}\left[\nu \sin x \sin y \sin z\right] + K^{-1}\left[\nu^{\alpha} K\left[\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right]\right]. \tag{5.26}$$

From equation (5.26),

$$u_0(x,y,z,t) = K^{-1}\left[v \sin x \sin y \sin z\right] = \sin x \sin y \sin z.$$

Now the recursive relation is given as

$$u_{n+1} = K^{-1}\left[v^{\alpha} K\left[\frac{\partial^2 u_n(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u_n(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u_n(x,y,z,t)}{\partial z^2}\right]\right]. \tag{5.27}$$

If $n = 0$ in equation (5.27), then

$$u_1(x,y,z,t) = K^{-1}\left[v^{\alpha} K\left[\frac{\partial^2 u_0(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u_0(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u_0(x,y,z,t)}{\partial z^2}\right]\right]. \tag{5.28}$$

Simplifying equation (5.28) gives

$$u_1(x,y,z,t) = K^{-1}\left[v^{\alpha} K\left[-3\sin x \sin y \sin z\right]\right] = K^{-1}\left[v^{\alpha}\left(-3v \sin x \sin y \sin z\right)\right]. \tag{5.29}$$

Then equation (5.29) becomes

$$u_1(x,y,z,t) = \left(-3\sin x \sin y \sin z\right) v^{\alpha} = \frac{-3t^{\alpha}}{\Gamma(\alpha+1)} \sin x \sin y \sin z.$$

If $n = 1$ in equation (5.16), then

$$u_2(x,y,z,t) = K^{-1}\left[v^{\alpha} K\left[\frac{\partial^2 u_1(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u_1(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u_1(x,y,z,t)}{\partial z^2}\right]\right]. \tag{5.30}$$

Simplifying equation (5.30) gives

$$u_2(x,y,z,t) = K^{-1}\left[v^{\alpha} K\left[\frac{9t^{\alpha}}{\Gamma(\alpha+1)} \sin x \sin y \sin z\right]\right]. \tag{5.31}$$

Further simplification of equation (5.31) gives

$$u_2(x,y,z,t) = K^{-1}\left[\frac{v^{\alpha}\alpha! t^{\alpha+1}}{\Gamma(\alpha+1)} 9 \sin x \sin y \sin z\right].$$

Then

$$u_2(x,y,z,t) = \frac{9t^{2\alpha}}{\Gamma(2\alpha+1)} \sin x \sin y \sin z.$$

Thus, the series solution becomes

$$u(x,y,z,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.$$

Then,

$$u(x,y,z,t) = \sin x \sin y \sin z - \frac{3t^{\alpha}}{\Gamma(\alpha+1)} \sin x \sin y \sin z + \frac{9t^{2\alpha}}{\Gamma(2\alpha+1)} \sin x \sin y \sin z + \cdots. \tag{5.32}$$

If $\alpha = 1$ in equation (5.32), then the close solution becomes

$$u(x,y,z,t) = e^{-3t} \sin x \sin y \sin z.$$

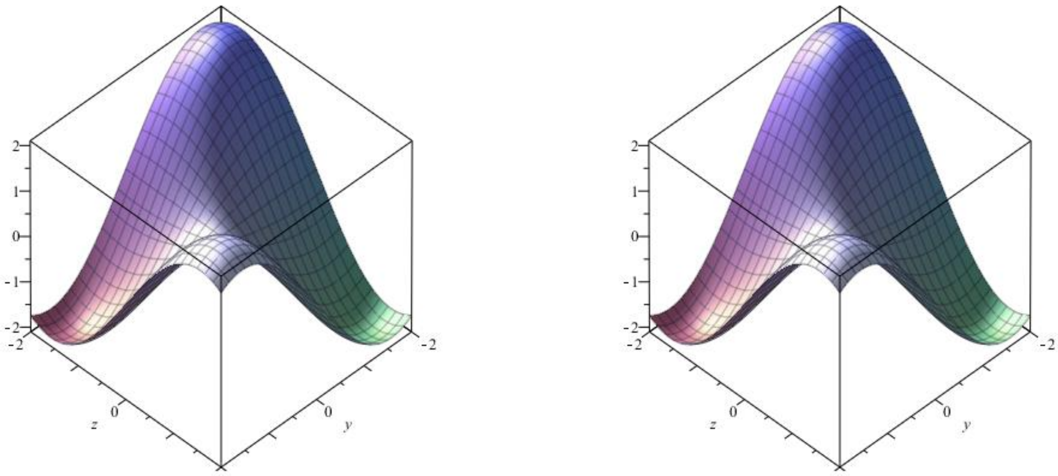


Figure 5: The left and right panel show the graphs of exact and Kamal transform method solution for $\alpha = 1$.

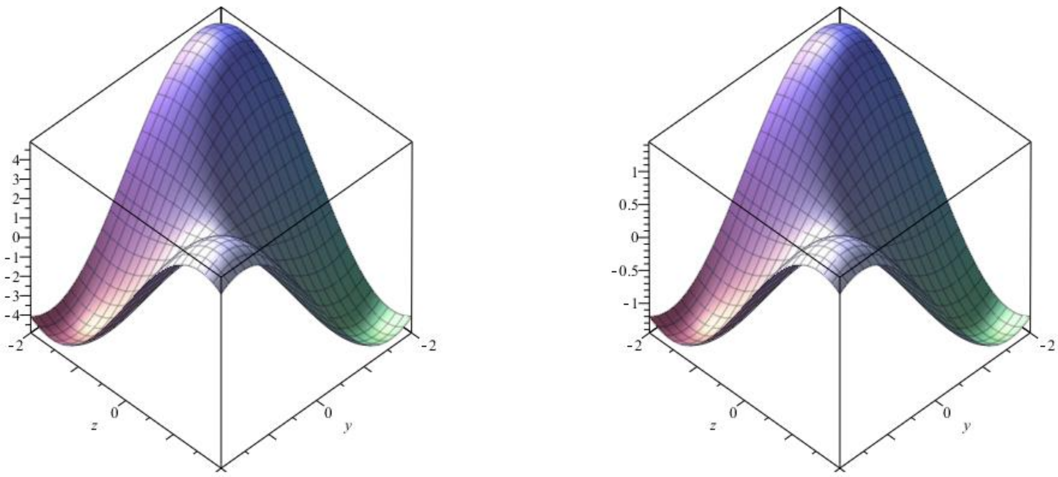


Figure 6: The left and right panel show the graphs of $\alpha = 0.2$ and $\alpha = 0.75$, respectively.

6. Discussion of results

The new scheme for solving fractional differential equations which are of Caputo form has been derived and the scheme was used to solve some set of time-linear fractional differential equations. It was observed that when $\alpha = 2$ and when $\alpha = 1$ in Problems 5.1, 5.2, and 5.3, respectively, there is a close solution between the exact and Kamal transform method.

Figures 1, 3, and 5 show the shape of the travelling wave when the differential equation is in classical form and it was observed that there is no change in the shape of the graph between that of the exact and the approximate solution.

Figures 2, 4, and 6 show the shape of the travelling wave at each derivative of α which are in fractional form, it was observed that at each derivative of α, the shape of the travelling wave changes as compared with the shape of the classical solution. The features of changes in shape the graphs at each derivative of α which are in form of fraction will help the scientists and engineers to know the model which fit in correctly in prediction stage.

7. Conclusion

In this article, we have proved how Kamal transform is used on Caputo derivative. A scheme was explained and analized, and the scheme was used to solve different set of Time-fractional differential equations. The solutions for certain illustrative problems are explained using the proposed scheme. The Kamal transform method result is in close contact with the exact solution of the suggested examples. The present scheme also calculate the solutions of the problems with the fractional order derivatives. Therefore, the proposed scheme can be extended to solve other complicated fractional order partial differential equations.

References

[1] K. Abdelilah, S. Hassan, *The new integral transform "Kamal Transform"*, Adv. Theor. Appl. Math., **11** (2016), 451–458. 3

[2] F. H. Ahmed, *Analytic approximate solutions for the 1D and 2D nonlinear fractional diffusion equations of Fisher type*, C. R. Acad. Bulgare Sci., **73**, (2020), 320–330. 1

[3] H. O. Al-Humedi, F. L. Hasan, *The Numerical Solutions of Nonlinear Time-Fractional Differential Equations by LMADM*, Iraq. J. Sci., **2021** (2021), 17–26. 1

[4] R. K. Bairwa, S. Karan, *Analytical Solution of Time-Fractional Klien-Gordon Equation by using Laplace-Adomian Decomposition Method*, Ann. Pure Appl. Math., **24** (2021), 27–35. 1, 4, 5.1

[5] M. S. H. Chowdhury, I. Hashim, *Application of homotopy-perturbation method to Klein-Gordon and sine-Gordon equations*, Chaos Solitons Fractals, **39** (2009), 1928–1935. 1

[6] Y. M. Chu, H. B. Ehab, R. E. Essam, E. Abdelhalim, A. S. Nehad, *Combination of Shehu decomposition and variational iteration transform methods for solving fractional third order dispersive partial differential equations*, Numerical Methods for Partial Differential Equations, **2021** (2021), 12 pages. 1

[7] S. Frassu, C. van der Mee, G. Viglialoro, *Boundedness in a nonlinear attraction-repulsion Keller–Segel system with production and consumption*, J. Math. Anal. Appl., **504** (2021), 20 pages. 1

[8] H. Gandhi, A. Tomar, D. Singh. *The Study of Linear and Nonlinear Fractional ODEs by Homotopy Analysis*, In: Soft Computing, **2022** (2022), 407–417.

[9] I. Hashim, O. Abdulaziz, S. Momani, *Homotopy analysis method for fractional IVPs*, Commun. Nonlinear Sci. Numer. Simul., **14** (2009), 674–684.

[10] H. Jafari, V. Daftardar-Gejji, *Solving linear and nonlinear fractional diffusion and wave equations by adomian decomposition*, Appl. Math. Comput., **180** (2006), 488–497. 1, 2.1

[11] H. Jafari, V. Daftardar-Gejji, *Solving a system of nonlinear fractional differential equations using Adomian decomposition*, J. Comput. Appl. Math., **196** (2006), 644–651. 1

[12] H. Jafari, C. M. Khalique, M. Nazari, *Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion–wave equations*, Appl. Math. Lett., **24** (2011), 1799–1805. 1

[13] H. Jafari, S. Seifi, *Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation*, Commun. Nonlinear Sci. Numer. Simul., **14** (2009), 2006–2012. 1

[14] A. Khalouta, A. Kadem, *A new method to solve fractional differential equations: Inverse fractional Shehu transform method*, Appl. Appl. Math., **14** (2019), 926–941. 3.2

[15] H. Khan, U. Farooq, R. Shah, D. Baleanu, P. Kumam, M. Arif, *Analytical solutions of (2+ time fractional order) dimensional physical models, using modified decomposition method*, Appl. Sci., **10** (2020), 12 pages. 5.2

[16] H. Khan, S. Rasool, K. Poom, B. Dumitru, A. Muhammad, *An efficient analytical technique, for the solution of fractional-order telegraph equations*, Mathematics, **7**, (2019), 14 pages. 1

[17] H. Khan, S. Rasool, K. Poom, B. Dumitru, A. Muhammad, *Laplace decomposition for solving nonlinear system of fractional order partial differential equations*, Adv. Difference Equ., **2020** (2020), 18 pages. 1

[18] T. X. Li, N. Pintus, G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys., **70** (2019), 18 pages. 1

[19] T. Li, G. Viglialoro, *Boundedness for a nonlocal reaction chemotaxis model even in the attraction dominated regime*, Differ. Integ. Equ., **34** (2021), 315–336. 1

[20] C. P. Li, Y. H. Wang, *Numerical algorithm based on Adomian decomposition for fractional differential equations*, Comput. Math. Appl., **57** (2009), 1672–1681. 1, 2.2

[21] S. T. Mohyud-Din, A. Yildirim, *Variational iteration method for solving Klein-Gordon equations*, J. Appl. Math. Stat. Inform., **6** (2010), 99–106. 1

[22] S. Momani, K. Al-Khaled, *Numerical solutions for systems of fractional differential equations by the decomposition method*, Appl. Math. Comput., **162** (2005), 1351–1365. 1

[23] S. Momani, Z. Odibat, *Numerical comparison of methods for solving linear differential equations of fractional order*, Chaos Solitons Fractals, **31** (2007), 1248–1255. 1

[24] S. Momani, O. Zaid, *Analytical solution of a time-fractional Navier-Stokes equation by Adomian decomposition method*, Appl. Math. Comput., **177** (2006), 488–494. 1, 2.1

[25] J. A. Owolabi, O. E. Ige, E. I. Akinola, *Application of Kamal Decomposition Transform Method in Solving Two Dimensional Unsteady Flow*, Int. J. Differ. Equ., **14** (2019), 207–214. 3, 3, 4

[26] J. A. Owolabi, R. A. Oderinu, *Kamal Transform Based Analytical Solution of a Generalized Nonlinear Hirota-Satsuma Coupled Equations*, Asian J. Pure Appl. Math., **2021** (2021), 20–29. 3, 4

[27] S. S. Ray, *A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends*, Appl. Math. Comput., **202** (2008), 544–549. 1, 2.2

[28] S. S. Ray, R. K. Bera, *Analytical solution of a fractional diffusion equation by Adomian decomposition method*, Appl. Math. Comput., **174** (2006), 329–336. 1

[29] R. Shah, K. Hassan, A. Muhammad, K. Poom, *Application of Laplace–Adomian decomposition method for the analytical solution of third-order dispersive fractional partial differential equations*, Entropy, **21** (2019), 17 pages. 1

[30] R. Shah, K. Hassan, K. Poom, A. Muhammad, *An analytical technique to solve the system of nonlinear fractional partial differential equations*, Mathematics, **7** (2019), 15 pages. 1

[31] K. Shah, H. Khalil, R. A. Khan, *Analytical solutions of fractional order diffusion equations by natural transform method*, Iran. J. Sci. Technol. Trans. A Sci., **42** (2018), 1479–1490. 5.3

[32] S. C. Sharma, R. K. Bairwa, *A reliable treatment of iterative Laplace transform method for fractional telegraph equations*, Ann. Pure Appl. Math., **9** (2015), 81–89. 1

[33] N. T. Shawagfeh, *Analytical approximate solutions for nonlinear fractional differential equations*, Appl. Math. Comput., **131** (2002), 517–529. 1

[34] P. Verma, K. Manoj. *An analytical solution of multi-dimensional space fractional diffusion equations with variable coefficients*, Int. J. Model. Simul. Scient. Comput., **12** (2021), 13 pages. 1

[35] G. Wu, E. W. M. Lee, *Fractional variational iteration method and its application*, Phys. Lett. A, **374** (2010), 2506–2509. 1

[36] M. Zurigat, *Solving nonlinear fractional differential equation using a multi-step Laplace Adomian decomposition method*, An. Univ. Craiova Ser. Mat. Inform., **39** (2012), 200–210. 1