A New Block Integrator for Second Order Initial Value Problems

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Abstract

A 4-step block integrator using Hermite polynomial as basis function for the solution of general second-order initial value problems is developed through interpolation and collocation procedures. The consistency, stability and convergence characteristics of the proposed methods are examined. Some linear and nonlinear test problems in literature are used for the numerical experimentation and the results obtained show the superiority of the method in comparison with some existing methods.

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1. Introduction

Modelling of several physical phenomena gives rise to second order differential equations. Examples of such phenomena are molecular dynamics, the mass movement under the action of a force, control theory, circuit theory control and chemical kinetics (see Singh and Ramos [12]). Consider a second order initial value problem of the form:

$$y'' = f(x,y,y'),\ y(a) = \alpha, \quad y'(a) = \beta, \qquad (1)$$

where f is a continuous differentiable function. The traditional approach of solving (1) in the literature is to first reduce it to an equivalent system of two first order initial value problems after which various applicable methods for the solution of first order initial value problems are used to solve the system of initial value problems obtained (see Areo and Adeniyi [4] and Jator [8]). However, in order to avoid complications associated with computer programs in the traditional approach, reduce the computer time and computational work, several authors, including Adeniran and Longe [2], Singla et al. [13], Ramos et al. [10] and Udo et al. [14], have proposed various methods for solving (1) directly without having to reduce it to a system of equations.

Block methods were formulated to overcome the demerits associated with predictor-corrector methods. Advantages of block methods include the fact that they are self-starting, efficient in terms of accuracy and pieces of solutions are not overlapped (see Ramos et al. [11]). This paper is concerned with the development of a block method through interpolation and collocation procedure for solving second-order ordinary differential equations using Hermite polynomial as basis functions.

The arrangement of this paper is as follows: In section 2, we develop a 4-step block method for solving (1) using Hermite polynomial as a basis function. Convergence analysis of the proposed method is established in section 3. In section 4, some test problems are solved to demonstrate the performance and reliability of the new method, and in the last section some conclusions are given.

2. Formulation of the Method

In order to solve the initial value problem (1) in

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the interval $[a,b]$ based on the partition $a = x_0 \leq x_1 \leq \cdots x_n = b$ with a uniform step length of $h = x_n - x_{n-1}$, $n = 0,1,\cdots,N-1$, we consider an approximate solution $Y(x)$ to the analytical solution $y(x)$ of the form:

$$Y(x) = \sum_{r=0}^{6} a_r H_r(x - x_n) \quad (2)$$

where $H_r(x)$ is the Probabilist's Hermite polynomial function of degree r defined as follows:

$$H_r(x) = (-1)^r e^{\frac{x^2}{2}} \frac{d^r}{dx^r} e^{-\frac{x^2}{2}}. \quad (3)$$

The Hermite polynomial (3) satisfies the recurrence relation:

$$H_{r+1}(x) = xH_r(x) - H_r'(x), \quad r \geq 1; \quad (4)$$

with the initial conditions: $H_0(x) = 1$ and $H_1(x) = x$.

To determine the values of the coefficients a_r, $r = 0(1)6$, in equation (2), set

$$\sum_{r=0}^{6} a_r H_r(x - x_n) = y_{n+i}, i = 1,3, \quad (5)$$

and

$$\sum_{r=0}^{6} a_r H_r''(x - x_n) = f_{n+i}, i = 0(1)4, \quad (6)$$

where n is the grid index and $y_{n+i} = Y(x_{n+i})$.

Equations (5) and (6) provide a system of seven equations whose solution gives the values of the coefficients a_r, $r = 0(1)6$, which are substituted into (2) and after some algebraic manipulations yields a continuous scheme of the form:

$$Y(x) = \sum_{i=0}^{4} \alpha_i(x) y_{n+i} + h^2 \sum_{i=0}^{4} \beta_i(x) f_{n+i}, \quad (7)$$

where $\alpha_i(x)$, $\beta_i(x)$, $i = 0(1)4$ are continuous coefficients.

Evaluating (7) at $x = x_n, x_{n+2}, x_{n+4}$ to obtain:

$$y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{240}(37f_n + 432f_{n+1} + 222f_{n+2} + 32f_{n+3} - 3f_{n+4}) \quad (8)$$

$$-y_{n+3} + 2y_{n+2} - y_{n+1} = \frac{h^2}{240}(f_n - 24f_{n+1} - 194f_{n+2} - 24f_{n+3} + f_{n+4}) \quad (9)$$

$$2y_{n+4} - 3y_{n+3} + y_{n+1} = \frac{h^2}{240}(-3f_n + 32f_{n+1} + 222f_{n+2} + 432f_{n+3} + 37f_{n+4}) \quad (10)$$

Now, differentiating (7) with respect to x to get

$$Y'(x) = \sum_{i=0}^{4} \alpha_i'(x) y_{n+i} + h^2 \sum_{i=0}^{4} \beta_i'(x) f_{n+i}, \quad (11)$$

Evaluating equation (11) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ to get the complementary methods as follows:

$$-y_{n+3} + y_{n+1} + 2hy_n' = \frac{h^2}{360}(-239f_n - 918f_{n+1} - 192f_{n+2} - 106f_{n+3} + 15f_{n+4}) \quad (12)$$

$$-y_{n+3} + y_{n+1} + 2hy_{n+1}' = \frac{h^2}{90}(3f_n - 68f_{n+1} - 114f_{n+2} - f_{n+4}) \quad (13)$$

$$-y_{n+3} + y_{n+1} + 2hy_{n+2}' = \frac{h^2}{360}(-7f_n + 74f_{n+1} - 74f_{n+3} + 7f_{n+4}) \quad (14)$$

$$-y_{n+3} + y_{n+1} + 2hy_{n+3}' = \frac{h^2}{90}(f_n + 114f_{n+1} + 68f_{n+2} - 3f_{n+4}) \quad (15)$$

$$-y_{n+3} + y_{n+1} + 2hy_{n+4}' = \frac{h^2}{360}(-15f_n + 106f_{n+1} + 192f_{n+2} + 918f_{n+3} + 239f_{n+4}) \quad (16)$$

The system of equations (8) –(10), (12)– (16) constitutes the new block method.

3. Analysis of the Method

Basic properties of the new block method are investigated to establish the efficiency and reliability of the method. The properties examined are: Order, error constant, consistence and zero stability. Now, the system of equations (8) –(10), (12)– (16) may be rewritten in matrix form as follows:

$$AY_m = BY_{m-1} + h^2[CF_{m-1} + DF_m], \quad (17)$$

where

$$Y_m = \left(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, hy_{n+1}', hy_{n+2}', hy_{n+3}', hy_{n+4}'\right)^T,$$

$$Y_{m-1} = \left(y_{n-3}, y_{n-2}, y_{n-1}, y_n, hy_{n-3}', hy_{n-2}', hy_{n-1}', hy_n'\right)^T,$$

$$F_m = \left(f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, hf_{n+1}', hf_{n+2}', hf_{n+3}', hf_{n+4}'\right)^T,$$

$$F_{m-1} = \left(f_{n-3}, f_{n-2}, f_{n-1}, f_n, hf_{n-3}', hf_{n-2}', hf_{n-1}', hf_n'\right)^T,$$

(where $f'_{n\pm i}, i = 1(1)4$ are introduced to augment the zero entries of the vector notations)

$$A=\begin{pmatrix} -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{9}{5} & \frac{37}{40} & \frac{2}{15} & -\frac{1}{80} & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{97}{120} & -\frac{1}{10} & \frac{1}{240} & 0 & 0 & 0 & 0 \\ \frac{2}{15} & \frac{37}{40} & \frac{9}{5} & \frac{37}{240} & 0 & 0 & 0 & 0 \\ -\frac{51}{20} & -\frac{8}{15} & -\frac{53}{180} & \frac{1}{24} & 0 & 0 & 0 & 0 \\ -\frac{34}{45} & -\frac{19}{15} & 0 & -\frac{1}{90} & 0 & 0 & 0 & 0 \\ \frac{37}{180} & 0 & -\frac{37}{180} & \frac{7}{360} & 0 & 0 & 0 & 0 \\ \frac{19}{15} & \frac{34}{45} & 0 & -\frac{1}{30} & 0 & 0 & 0 & 0 \\ \frac{53}{180} & \frac{8}{15} & \frac{51}{20} & \frac{239}{360} & 0 & 0 & 0 & 0 \end{pmatrix}. \quad D=\begin{pmatrix} 0 & 0 & 0 & \frac{37}{240} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{240} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{80} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{239}{360} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{30} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{37}{180} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{90} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{24} & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.1 Local Truncation Error

According to Jator [8] and Lambert [9], the local truncation error associated with the second order linear multistep method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f_{n+i} \tag{18}$$

is defined by the difference operator

$$L[y(x):h] = \sum_{i=0}^{k} [\alpha_i y(x_n + ih) - h^2 \beta_i f(x_n + ih)], \tag{19}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$.
Expanding (19) in Taylor series about point x leads to the expression:

$$L[y(x):h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) + c_{p+1} h^{p+1} y^{(p+1)}(x) + c_{p+2} h^{p+2} y^{(p+2)}(x), \tag{20}$$

where $C_0, C_1, C_2, \cdots, C_p, \cdots, C_{p+1}$ are obtained as follows:

$$C_0 = \sum_{i=0}^{k} \alpha_i, \quad C_1 = \sum_{i=1}^{k} i\alpha_i, \quad C_2 = \frac{1}{2!} \sum_{i=1}^{k} i^2 \alpha_i', \quad \cdots, \quad C_q = \frac{1}{q!}\left[\sum_{i=1}^{k} i^q \alpha_i - q(q-1) \sum_{i=1}^{k} \beta_i i^{q-2}\right] \tag{21}$$

Method (18) is said to be of order p if $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$. The constant $C_{p+2} \neq 0$ is called the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x)$ is the principal local truncation error at x_n.

Following from the definition above, the block method (17) is of order $p = (5,5,5,5,5,5,5,5)$ with error constant

$$C_{p+2} = \left(\frac{1}{120}, 0, \frac{-1}{120}, \frac{-149}{5040}, \frac{1}{126}, \frac{-37}{5040}, \frac{13}{315}, \frac{-149}{5040}\right)^T.$$

3.2 Consistency

Following Jator [8], the block method (17) is consistent since the condition $p \geq 1$ is satisfied.

3.3 Zero Stability

The normalization of the system (17) is given by

$$A^* Y_m = B^* Y_{m-1} + h^2 \left[C^* F_{m-1} + D^* F_m\right], \tag{22}$$

and as $h \to 0$, (22) tends to the difference system

$$A^* Y_m - B^* Y_{m-1} = 0, \tag{23}$$

where A^* is an identity matrix of order 8 and

$$B^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The first characteristic polynomial of (23) is given by

$$\rho(r) = \det\ rA^* - B^*) = r^6 (r-1)^2. \tag{24}$$

Therefore, the block method (17) is zero-stable since according to Fatula [6] a block method is zero-stable if $\rho(r) = 0$ satisfies $|r_i| \leq 1$ and the multiplicity of the roots with $|r_i| = 1$ does not exceed 2.

3.4 Convergence

By Dahlquist [5], the necessary and sufficient conditions for a linear multistep method to be convergent is to be consistent and zero stable. Therefore, the proposed block method converges.

4. Numerical Examples

To demonstrate the efficiency of the method, we apply the proposed method to solve the following linear and nonlinear test problems from literature:

Example 1

Consider the following linear initial value problem:

$y'' - 100y = 0; \quad y(0) = 1, y'(0) = -10, h = 0.01.$

Analytical solution: $y(x) = e^{-10x}$

Source: Adee et al. [1] and Areo and Adeniyi [4].

Example 2

Solve the initial value problem:

$y'' + y = 0; y(0) = 1, y'(0) = 1, h = 0.1$

Analytical solution: $y(x) = \cos x + \sin x.$

Source: Adee et al. [1] and Areo and Adeniyi [4].

Example 3

Solve the initial value problem:

$y'' - y' = 0; y(0) = 0, y'(0) = -1, h = 0.1.$

Analytical solution: $y(x) = 1 - e^x.$

Source: Yahaya and Badmus [15].

Example 4

Solve:

$y'' + (y')^2 + y^2 = 1 - \sin x; h = 0.1, 0 \le x \le 1.$

subject to the initial conditions: $y(0) = 0, y'(0) = 1.$

Analytical solution: $y(x) = \sin x.$
Source: Guler et al. [7].

Example 5

Solve the nonlinear initial value problem:

$$y'' - 2y^3 = 0; \quad y(1) = 1, \quad y'(1) = -1, \quad h = \frac{0.1}{40}$$

Analytical solution: $y(x) = \frac{1}{x}.$

Source: Adesanya et al. [3]

Table 1: Absolute Errors for Example 1

x	New Method	Adee et al. [1]	Areo and Adeniyi [4]
0.01	8.96E-10	4.96E-09	3.00E-10
0.02	2.16E-09	1.49E-08	1.00E-09
0.03	3.43E-09	6.68E-09	2.80E-09
0.04	4.42E-09	2.40E-08	3.00E-09
0.05	5.19E-09	6.78E-07	5.10E-09
0.06	6.26E-09	1.08E-07	6.40E-09
0.07	7.39E-09	1.77E-07	9.40E-09
0.08	8.37E-09	2.32E-07	1.15E-08
0.09	9.27E-09	3.13E-07	1.49E-08
0.10	1.04E-08	3.95E-07	1.77E-08
0.11	1.17E-08	4.96E-07	2.30E-08
0.12	1.29E-08	6.00E-06	2.70E-08

5. Conclusion

In this paper, we have constructed a direct 4–step block integrator which is suitable and efficient for solving second order initial value problems in ordinary differential equations. The analysis of the properties of the proposed method show that the method is zero stable, consistent and convergent. The method is applied on some linear and nonlinear test problems with favourable solutions obtained.

Table 2: Absolute Errors for Example 2

x	New Method	Adee et al. [1]	Areo and Adeniyi [4]
0.1	8.61E-10	6.92E-09	0.0
0.2	2.06E-09	1.76E-08	6.0E-09
0.3	3.23E-09	1.62E-08	1.8E-08
0.4	4.06E-09	4.37E-08	2.5E-08
0.5	4.37E-09	1.20E-07	3.4E-08
0.6	4.77E-09	1.87E-07	4.7E-08
0.7	5.12E-09	3.07E-07	5.3E-08
0.8	5.32E-09	4.19E-07	6.4E-08
0.9	4.97E-09	5.79E-07	7.5E-08
1.0	4.45E-09	7.27E-07	8.8E-08
1.1	3.89E-09	9.20E-07	9.8E-08
1.2	3.43E-09	1.10E-06	1.11E-07

Table 3: Absolute Errors for Example 3

x	New Method	Yahaya and Badmus [15]	Yahaya and Badmus [15]
		(Optimal Method)	(Hybrid Method)
0.1	1.34E-09	5.01E-03	8.79E-05
0.2	2.40E-09	1.10E-02	3.27E-04
0.3	5.65E-09	1.90E-02	2.22E-03
0.4	7.63E-09	2.84E-02	4.86E-03
0.5	1.04E-08	4.00E-02	9.10E-03
0.6	1.44E-08	5.34E-02	1.44E-02
0.7	1.87E-08	6.95E-02	2.14E-02
0.8	2.28E-08	8.77E-02	2.99E-02
0.9	2.82E-08	1.09E-01	4.03E-02
1.0	3.54E-08	1.33E-01	5.26E-02

Table 4: Absolute Errors for Example 4

x	New Method	Guler et al. [7] (N=5)
0.1	9.30E-10	3.49E-08
0.2	1.99E-09	1.16E-07
0.4	3.18E-09	1.13E-07
0.5	3.23E-09	4.61E-07
0.6	3.51E-09	7.80E-07
0.7	3.74E-09	1.40E-06
0.8	3.53E-09	4.16E-06
0.9	3.03E-09	1.40E-05
1.0	2.75E-09	4.10E-05

Table 5: Absolute Errors for Example 5

x	New Method	Adesanya et al. [3]
1.0025	3.16E-17	4.56E-07
1.0050	7.55E-17	1.94E-06
1.0075	1.19E-16	8.40E-06
1.0100	1.51E-16	2.81E-05
1.0125	1.81E-16	6.14E-05
1.0150	2.22E-16	9.47E-05
1.0175	2.63E-16	1.68E-04
1.0200	2.93E-16	2.37E-04
1.0225	3.21E-16	3.07E-04
1.0250	3.59E-16	4.15E-04

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