International Journal of Difference Equations (IJDE).
ISSN 0973-6069, Volume 17, Number 2, (2022). 255-266

http://www.ripublication.com/ijde.htm

Effective Solution Method for the Second Kind Volterra Integral Equations with Separable Kernels

O.M. Ogunlaran[1] **and N.K. Oladejo** [2]

[1]*Mathematics Programme, College of Agriculture, Engineering and Science, Bowen University, Nigeria.* [1]*Email: matthew.ogunlaran@bowen.edu.ng*
[2]*Department of Industrial Mathematics, C. K. Tedam University of Technology and Applied Sciences, Ghana.* [2]*Email: noladejo@cktutas.edu.gh*

Abstract

This paper examined the solution of Volterra integral equations of the second kind with separable kernels using Taylor series expansion method. A simple formula for expressing the derivatives of the unknown function in terms of lower order derivatives is derived. The proposed method is applied to solve both cases of linear and non linear Volterra integral equations of the second kind with separable kernels obtained from the literature. The results of the numerical examples considered showed the efficiency and reliability of the method.

Keywords: Taylor series, Volterra integral equations, separable kernels.

1. INTRODUCTION

The importance of Volterra integral equations cannot be overemphasized in scientific and engineering applications. Volterra integral equations are encountered in mathematical modeling of population dynamics, epidemiology and semi-conductor devices [18]. The equations also arise as a result of reformulation of initial value problems.

In recent times, attention has been given to the development of analytical and approximate approaches for the solution of both linear and nonlinear forms of these integral equations. Adomian [1] proposed Adomian decomposition method for solving linear and non linear integral equations. Costerell and spigler [4] applied signoidal function to develop a method for tackling Volterra integral equations of the second kind. Wazwaz [18, 19] proposed Variational Iteration method, Adomian decomposition method, Laplace transform method and successive approximation method for solving Fredholm and Volterra integral equations of the first and second kinds. Wazwaz [20, 21] developed a modified Adomian decomposition method for differential equation and linear and non linear integral equations. Brunner [3] suggested implicitly linear collocation method to solve non linear Volterra integral equations and integro-differential equations with Hemmerstein kernel. Tang et al. [17] studied product integration method to solve integral equations with logarithmic singular kernel while Diogo et al.[16] considered Hermite type collocation method for the solution of second kind Volterra integral equations with a certain weakly singular kernel. Ogunlaran and Akinlotan [12] developed a spline collocation method for solving system of linear Fredholm integral equations. Ogunlaran and Oke [13] examined a spline collocation method for the solution of first order integro-differential equations. Odibat [11] applied Differential transform method for the solution of Volterra integral equations of the second kind with separable kernel. Lichea et al. [7] proposed Runge-Kutta methods for solving the second kind Volterra integral equations with singular kernel. Rocha et al. [15] formulated a collocation method with piecewise continuous basis functions to solve integral equations of the second kind.

Ren et al. [14] and Maleknejad et al. [9] proposed Taylor series method for solving Fredholm integral equations of the second kind with weakly smooth or weakly singular kernel while Huabsomboon [5] in an attempt to improve on the results obtained in [14] presented a modified form of Taylor series method to solve the same problem with better results obtained. Majeknejad and Aghazadeh [9] applied Taylor series method for the solution of the Volterra integral equations of the second kind with weakly smooth or weakly singular kernel. However, in order to improve on the results in [8], Huabsomboon et al. [6] also studied Taylor series method. Similarly, Majeknejad and Damencheli [10] and Barikbin [2] proposed Taylor series method for Volterra integral equations with convolution kernels. Despite the successful applications of Taylor series method for solving a wide range of equations by several authors, one major drawback of the method generally is the difficulty posed in obtaining the derivatives of the unknown function, especially if the function is complicated.

The second kind Volterra integral equation is an equation of the form

$$y(x) = f(x) + \int_0^x k(x,t)y(t)dt, \tag{1}$$

where $f(x)$ is a known function, $k(x,t)$ is a known integral kernel and $y(x)$ is the unknown function to be determined.
The focus of this work is to employ Taylor series method for the solution the second kind Volterra integral equations with separable kernels, that is

$$k(x,t) = \sum_{i=1}^{m} g_i(x)h_i(t), \tag{2}$$

such that the second order Volterra integral equation (1) can be written as

$$y(x) = f(x) + \sum_{i=1}^{m} g_i(x) \int_0^x h_i(t)y(t)dt. \tag{3}$$

An approach has also been derived in this study to circumvent the drawback associated with differentiation mentioned above during the process of solving Volterra integral equations with separable kernels using Taylor series method.

2. METHOD OF SOLUTION

Theorem 2.1. Let $y(x)$ be defined as in (3) and suppose $y(x)$ is a smooth real-valued function of a real variable, then the jth order derivative of $y(x)$ is given by

$$\begin{aligned} y^{(j)}(x) = f^{(j)}(x) + \sum_{k=0}^{j-1}\Bigg[& \binom{j}{1}\binom{0}{k}\sum_{i=1}^{m} g_i^{(j-1)}(x)h_i^{(0)}(x) + \binom{j}{2}\binom{1}{k}\sum_{i=1}^{m} g_i^{(j-2)}(x)h_i^{(1-k)}(x) \\ & + \binom{j}{3}\binom{2}{k}\sum_{i=1}^{m} g_i^{(j-3)}(x)h_i^{(2-k)}(x) + \cdots + \binom{j}{j-1}\binom{j-2}{k}\sum_{i=1}^{m} g_i'(x)h_i^{(j-k-2)}(x) \\ & + \binom{j}{j}\binom{j-1}{k}\sum_{i=1}^{m} g_i^{(0)}(x)h_i^{(j-k-1)}(x)\Bigg] y^{(k)}(x) + \sum_{i=1}^{m} g_i^{(j)}(x)\int_0^x h_i(t)y(t)dt. \end{aligned} \tag{4}$$

Proof. The proof of Theorem 2.1 shall be established by mathematical induction.
For $j = 1$, that is p_1, we obtain from the theorem 2.1.

$$y'(x) = f'(x) + \binom{1}{1}\binom{0}{0}\sum_{i=1}^{m} g_i(x)h_i(x)y(x) + \sum_{i=1}^{m} g_i'(x)\int_0^x h_i(t)y(t)dt. \tag{5}$$

or

$$y'(x) = f'(x) + \sum_{i=1}^{m} g_i(x)h_i(x)y(x) + \sum_{i=1}^{m} g_i'(x)\int_0^x h_i(t)y(t)dt, \tag{6}$$

which is true.
Suppose that p_r holds, that is

$$
\begin{aligned}
y^{(r)}(x) = f^{(r)}(x) + \sum_{k=0}^{r-1} \Bigg[\binom{r}{1}\binom{0}{k} \sum_{i=1}^{m} g_i^{(r-1)}(x) h_i^{(0)}(x) + \binom{r}{2}\binom{1}{k} \sum_{i=1}^{m} g_i^{(r-2)}(x) h_i^{(1-k)}(x) \\
+ \binom{r}{3}\binom{2}{k} \sum_{i=1}^{m} g_i^{(r-3)}(x) h_i^{(2-k)}(x) + \cdots + \binom{r}{r-1}\binom{r-2}{k} \sum_{i=1}^{m} g_i'(x) h_i^{(r-k-2)}(x) \\
+ \binom{r}{r}\binom{r-1}{k} \sum_{i=1}^{m} g_i^{(0)}(x) h_i^{(r-k-1)}(x) \Bigg] y^{(k)}(x) + \sum_{i=1}^{m} g_i^{(r)}(x) \int_0^x h_i(t) y(t) dt. \qquad (7)
\end{aligned}
$$

We will now establish the validity of p_{r+1} based on the induction step (7). Now,

$$
y^{(r+1)} = \frac{d}{dx}\left(y^{(r)} \right). \qquad (8)
$$

Substituting equation (7) into (8) yields

$$
\begin{aligned}
y^{(r+1)}(x) = \frac{d}{dx}\Bigg(f^{(r)}(x) + \sum_{k=0}^{r-1} \Bigg[\binom{r}{1}\binom{0}{k} \sum_{i=1}^{m} g_i^{(r-1)}(x) h_i^{(0)}(x) + \binom{r}{2}\binom{1}{k} \sum_{i=1}^{m} g_i^{(r-2)}(x) h_i^{(1-k)}(x) \\
+ \binom{r}{3}\binom{2}{k} \sum_{i=1}^{m} g_i^{(r-3)}(x) h_i^{(2-k)}(x) + \cdots + \binom{r}{r-1}\binom{r-2}{k} \sum_{i=1}^{m} g_i'(x) h_i^{(r-k-2)}(x) \\
+ \binom{r}{r}\binom{r-1}{k} \sum_{i=1}^{m} g_i^{(0)}(x) h_i^{(r-k-1)}(x) \Bigg] y^{(k)}(x) + \sum_{i=1}^{m} g_i^{(r)}(x) \int_0^x h_i(t) y(t) dt \Bigg). \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
y^{(r+1)}(x) = f^{(r+1)}(x) + \sum_{k=0}^{r-1} \Bigg[\binom{r}{1}\binom{0}{k} \sum_{i=1}^{m} \left(g_i^{(r)}(x) h_i^{(0)}(x) + g_i^{(r-1)}(x) h_i'(x) \right) \\
+ \binom{r}{2}\binom{1}{k} \sum_{i=1}^{m} \left(g_i^{(r-1)}(x) h_i^{(1-k)}(x) + g_i^{(r-2)}(x) h_i^{(2-k)}(x) \right) \\
+ \binom{r}{3}\binom{2}{k} \sum_{i=1}^{m} \left(g_i^{(r-2)}(x) h_i^{(2-k)}(x) + g_i^{(r-3)}(x) h_i^{(3-k)}(x) \right) + \cdots \\
+ \binom{r}{r-1}\binom{r-2}{k} \sum_{i=1}^{m} \left(g_i''(x) h_i^{(r-k-2)}(x) + g_i'(x) h_i^{(r-k-1)}(x) \right) \\
+ \binom{r}{r}\binom{r-1}{k} \sum_{i=1}^{m} \left(g_i'(x) h_i^{(r-k-1)}(x) + g_i^{(0)}(x) h_i^{(r-k)}(x) \right) \Bigg] y^{(k)}(x) \\
+ \sum_{k=0}^{r-1} \Bigg[\binom{r}{1}\binom{0}{k} \sum_{i=1}^{m} g_i^{(r-1)}(x) h_i^{(0)}(x) + \binom{r}{2}\binom{1}{k} \sum_{i=1}^{m} g_i^{(r-2)}(x) h_i^{(1-k)}(x) \qquad (10)
\end{aligned}
$$

$$+\binom{r}{3}\binom{2}{k}\sum_{i=1}^{m}g_i^{(r-3)}(x)h_i^{(2-k)}(x)+\cdots+\binom{r}{r-1}\binom{r-2}{k}\sum_{i=1}^{m}g_i'(x)h_i^{(r-k-2)}(x)$$
$$+\binom{r}{r}\binom{r-1}{k}\sum_{i=1}^{m}g_i^{(0)}(x)h_i^{(r-k-1)}(x)\Bigg]y^{(k+1)}(x)+\sum_{i=1}^{m}g_i^{(r)}(x)h_i(x)y(x)$$
$$+\sum_{i=1}^{m}g_i^{(r+1)}(x)\int_0^x h_i(t)y(t)dt. \tag{11}$$

$$y^{(r+1)}(x)=f^{(r+1)}(x)+\sum_{k=0}^{r-1}\Bigg[\binom{r}{1}\binom{0}{k}\sum_{i=1}^{m}\left(g_i^{(r)}(x)h_i^{(0)}(x)+g_i^{(r-1)}(x)h_i'(x)\right)$$
$$+\binom{r}{2}\binom{1}{k}\sum_{i=1}^{m}\left(g_i^{(r-1)}(x)h_i^{(1-k)}(x)+g_i^{(r-2)}(x)h_i^{(2-k)}(x)\right)$$
$$+\binom{r}{3}\binom{2}{k}\sum_{i=1}^{m}\left(g_i^{(r-2)}(x)h_i^{(2-k)}(x)+g_i^{(r-3)}(x)h_i^{(3-k)}(x)\right)+\cdots$$
$$+\binom{r}{r-1}\binom{r-2}{k}\sum_{i=1}^{m}\left(g_i''(x)h_i^{(r-k-2)}(x)+g_i'(x)h_i^{(r-k-1)}(x)\right)$$
$$+\binom{r}{r}\binom{r-1}{k}\sum_{i=1}^{m}\left(g_i'(x)h_i^{(r-k-1)}(x)+g_i^{(0)}(x)h_i^{(r-k)}(x)\right)\Bigg]y^{(k)}(x)$$
$$+\sum_{k=0}^{r-1}\Bigg[\binom{r}{1}\binom{0}{k}\sum_{i=1}^{m}g_i^{(r-1)}(x)h_i^{(0)}(x)+\binom{r}{2}\binom{1}{k}\sum_{i=1}^{m}g_i^{(r-2)}(x)h_i^{(1-k)}(x)$$
$$+\binom{r}{3}\binom{2}{k}\sum_{i=1}^{m}g_i^{(r-3)}(x)h_i^{(2-k)}(x)+\cdots+\binom{r}{r-1}\binom{r-2}{k}\sum_{i=1}^{m}g_i'(x)h_i^{(r-k-2)}(x)$$
$$+\binom{r}{r}\binom{r-1}{k}\sum_{i=1}^{m}g_i^{(0)}(x)h_i^{(r-k-1)}(x)\Bigg]y^{(k+1)}(x)+\sum_{i=1}^{m}g_i^{(r)}(x)h_i(x)y(x)$$
$$+\sum_{i=1}^{m}g_i^{(r+1)}(x)\int_0^x h_i(t)y(t)dt. \tag{12}$$

Simplifying, we obtain

$$y^{(r+1)}(x)=f^{(r+1)}(x)+\sum_{k=0}^{r}\Bigg[\binom{r+1}{1}\binom{0}{k}\sum_{i=1}^{m}g_i^{(r)}(x)h_i^{(0)}(x)+\binom{r+1}{2}\binom{1}{k}\sum_{i=1}^{m}g_i^{(r-1)}(x)h_i^{(1-k)}(x)$$
$$+\binom{r+1}{3}\binom{2}{k}\sum_{i=1}^{m}g_i^{(r-2)}(x)h_i^{(2-k)}(x)+\cdots+\binom{r+1}{r}\binom{r-2}{k}\sum_{i=1}^{m}g_i'(x)h_i^{(r-k-1)}(x)$$
$$+\binom{r+1}{r+1}\binom{r}{k}\sum_{i=1}^{m}g_i^{(0)}(x)h_i^{(r-k)}(x)\Bigg]y^{(k)}(x)+\sum_{i=1}^{m}g_i^{(r+1)}(x)\int_0^x h_i(t)y(t)dt, \tag{13}$$

which is true. Thus, the theorem is true. □

Table 1: Coefficients of $y^{(k)}(x)$ in $y^{(j)}(x)$

	y	y'	y''	$\cdots$	$y^{(j-2)}$	$y^{(j-1)}$
$\binom{j}{1}$	$\binom{0}{0}g^{(j-1)}h$					
$\binom{j}{2}$	$\binom{1}{0}g^{(j-2)}h'$	$\binom{1}{1}g^{(j-2)}h$				
$\binom{j}{3}$	$\binom{2}{0}g^{(j-3)}h''$	$\binom{2}{1}g^{(j-3)}h'$	$\binom{2}{2}g^{(j-3)}h$			
$\binom{j}{4}$	$\binom{3}{0}g^{(j-4)}h'''$	$\binom{3}{1}g^{(j-4)}h''$	$\binom{3}{2}g^{(j-4)}h'$			
$\vdots$	$\vdots$	$\vdots$		$\cdots$		
$\binom{j}{j-1}$	$\binom{j-2}{0}g'h^{(j-2)}$	$\binom{j-2}{1}g'h^{(j-3)}$	$\binom{j-2}{2}g'h^{(j-4)}$		$\binom{j-2}{j-2}g'h$	
$\binom{j}{j}$	$\binom{j-1}{0}gh^{(j-1)}$	$\binom{j-1}{1}gh^{(j-2)}$	$\binom{j-1}{2}gh^{(j-3)}$		$\binom{j-1}{j-2}gh'$	$\binom{j-1}{j-1}gh$

For the case $m = 1$, $k(x,t) = g_1(x)h_1(t)$. However, let $g_1(x) \equiv g(x)$ and $h_1(x) \equiv h(t)$ such that $k(x,t) = g(x)h(t)$, then (4) reduces to:

$$y^{(j)}(x) = f^{(j)}(x) + \sum_{k=0}^{j-1}\left[\binom{j}{1}\binom{0}{k}g^{(j-1)}h^{(0)} + \binom{j}{2}\binom{1}{k}g^{(j-2)}h^{(1-k)} + \binom{j}{3}\binom{2}{k}g^{(j-3)}h^{(2-k)} + \cdots + \binom{j}{j-1}\binom{j-2}{k}g'h^{(j-k-2)} + \binom{j}{j}\binom{j-1}{k}gh^{(j-k-1)}\right]y^{(k)}(x) + g^{(j)}(x)\int_0^x h(t)y(t)dt. \tag{14}$$

The derivatives of y (i.e. $y^{(j)}$), as given by equation (14) (and likewise by equation (4)), are made up of three components which are namely: $f^{(j)}(x)$, $g^{(j)}(x)\int_0^x h(t)y(t)dt$ and the expressions involving $y^{(k)}(x), k = 0, 1, \cdots, j-1$. It was observed that the coefficient of each of $f^{(j)}(x)$ and $g^{(j)}(x)\int_0^x h(t)y(t)dt$ is 1. However, the coefficients of $y^{(k)}(x)$ in equation (14) can be written in tabular form as in Table 1.

Note that the first entry in each row multiplies every entry in the row and then the summation of the entries in each column gives the corresponding expression involving $y^{(k)}(x)$. We observed from equation (14) after leaving out $f^{(j)}(x)$ and $g^{(j)}(x)\int_0^x h(t)y(t)dt$ or directly from table 1 as follows:

(i) $y^{(j)}(x)$ is expressed in terms of j lower derivatives of y. That is, $y^{(k)}, k = 0, 1, \cdots, (j-1)$;

(ii) addition of the orders of the derivatives of g, h and y for each term of $y^{(j)}(x)$ equals $(j-1)$;

(iii) there are $(j-k)$ terms involving each of the lower derivatives $y^{(k)}(x)$, $k =$

Table 2: Coefficients of $y^{(k)}(x)$ in $y'''(x)$

	y	y'	y''
3	1		
3	1	1	
1	1	2	1

Table 3: Coefficients of $y^{(k)}(x)$ in $y^{(5)}(x)$

	y	y'	y''	y'''	$y^{(4)}$
5	1				
10	1	1			
10	1	2	1		
5	1	3	3	1	
1	1	4	6	4	1

$0, 1, \cdots, j-1$ of $y^{(j)}(x)$. For instance, in $y^{(5)}(x)$ there are 5 terms involving y, 4 terms involving y', $\cdots$ and 1 term involving $y^{(\prime v)}(x)$.

(iv) the order of derivative of g is $(j-k-1)$ in the first term of $y^{(k)}(x)$ and it decreases by one in each successive term till it becomes 0. On the other hand, the order of derivative of h increases from zero by 1 till it becomes $(j-k-1)$. For instance, for $y^{(5)}(x)$, the products of derivatives of the functions involved for y are $g^{\prime v}h$, $g'''h'$, $g''h''$, $\cdots$, $gh^{\prime v}$.

Now excluding the functions g, h and their derivatives from Table 1, we obtain for particular cases $n = 3, 5, 7$ as given in Tables $2-4$.

0 We further observed that:

Table 4: Coefficients of $y^{(k)}(x)$ in $y^{(7)}(x)$

	y	y'	y''	y'''	$y^{(4)}$	$y^{(5)}$	$y^{(6)}$
7	1						
21	1	1					
35	1	2	1				
35	1	3	3	1			
21	1	4	6	4	1		
7	1	5	10	10	5	1	
1	1	6	15	20	15	6	1

(v) leaving out the first row and column in Tables $2-4$, the remaining entries form a Pascal triangle.

(vi) the entries on the first column of each of the tables correspond to the entries that would be in the next row if the Pascal triangle were to be extended by one row but with the first element ignored.

(vii) the coefficients of $y^{(k)}$ in equation (14) can be expressed exactly as in Tables $1-4$. Note that Tables 2-4 exclude the products of derivatives of $g_i(x)$ and $h_i(t)$.

The Volterra integral equation of the second kind is estimated by using the nth order Taylor polynomial of degree n about the point x_0 given as follows:

$$y(x) = \sum_{k=0}^{n} \frac{y^{(j)}(x_0)}{j!}(x-x_0)^j, \tag{15}$$

where $y^{(j)}(x)$ is as defined in (4).

The error bound associated with the approximate solution (15) for the solution of the Volterra integral equation of the second kind (1) is given by:

$$|R_{n+1}(x)| \leq \frac{M}{(n+1)!}y^{(n+1)}(\xi) \quad \xi \in [x_0, x], \tag{16}$$

where $M = max|(x-x_0)^{n+1}|$.

3. TEST PROBLEMS

In this section, the proposed method is applied to solve some test problems about $x_0 = 0$ in order to demonstrate the efficiency of the method for handling the second kind Volterra integral equation with separable kernels.

Problem 3.1

Solve the Volterra integral equation

$$y(x) = x + \int_0^x (t-x)y(t)dt, \ 0 < x < 1. \tag{17}$$

The exact solution is $y(x) = \sin x$.

Source: Odibat [11].

Here, $f(x) = x$ and let $k(x,t) = g_1(x)h_1(t) + g_2(x)h_2(t)$, where $g_1(x) = 1$, $h_1(t) = t$, $g_2(x) = x$, $h_2(t) = -1$.

Applying Theorem 2.1, we obtain:

$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = -1, \quad u'^{v}(0) = 0,$
$y^{(5)}(0) = 1, \quad y^{(6)}(0) = 0, \quad y^{(7)}(0) = -1, \quad y^{(8)}(0) = 0, \quad y^{(9)}(0) = 1.$
Therefore following from (15), the solution of (17) is obtained as

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots$$

which is the closed form solution of the exact solution for the equation (17).

Problem 3.2
Solve

$$y(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x-t)y(t)dt, \ 0 < x < 1. \tag{18}$$

The exact solution is $y(x) = 1 - \sinh x$.
Source: Odibat [11].
In this case, $f(x) = 1 - x - \frac{x^2}{2}$ and $k(x,t) = g_1(x)h_1(t) + g_2(x)h_2(t)$. Choose $g_1(x) = x$, $h_1(t) = 1$, $g_2(x) = -1$, $h_2(t) = t$.
Applying Theorem 2.1, we obtain:
$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 0, \quad y'''(0) = -1, \quad y'^{v}(0) = 0,$
$y^{(5)}(0) = -1, \quad y^{(6)}(0) = 0, \quad y^{(7)}(0) = -1, \quad y^{(8)}(0) = 0, \quad y^{(9)}(0) = -1.$
Consequently, the solution of (18) is obtained as

$$y(x) = 1 - x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5 - \frac{1}{7!}x^7 - \frac{1}{9!}x^9 - \cdots$$

which is the closed form solution of the exact solution for the equation (18).

Problem 3.3
Consider the nonlinear Volterra integral equation

$$y(x) + \int_0^x \big(y^2(t) + y(t)\big)dt = \frac{3}{2} - \frac{1}{2}e^{(-2x)}, \ 0 < x < 1. \tag{19}$$

The exact solution is $y(x) = \exp(-x)$.
Source: Odibat [11].

In this case, $f(x) = \frac{3}{2} - \frac{1}{2}e^{(-2x)}$ and let $k(x,t) = g(x)h(t)$. Choose $g(x) = -1$, $h(t) = y(t) + 1$.
Applying Theorem 2.1, we obtain:
$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1, \quad y'''(0) = -1, \quad y'^{v}(0) = 1,$
$y^{(5)}(0) = -1, \quad y^{(6)}(0) = 1, \quad y^{(7)}(0) = -1, \quad y^{(8)}(0) = 1, \quad y^{(9)}(0) = -1.$

Consequently, the solution of (19) is
$y(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \cdots$
which is the closed form solution of the exact solution for the equation (19).

Problem 3.4
Solve the Volterra integral equation

$$y(x) = 1 - \int_0^x (x-t)y(t)dt,\ 0 < x < 1. \tag{20}$$

The exact solution is $y(x) = \cos x$.
Source: Odibat [11].
In this case, $f(x) = 1$ and $k(x,t) = g_1(x)h_1(t) + g_2(x)h_2(t)$, where $g_1(x) = x$, $h_1(t) = -1$, $g_2(x) = 1$, $h_2(t) = t$. Applying Theorem 2.1, we obtain:
$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0, \quad y'^v(0) = 1,$
$y^{(5)}(0) = 0, \quad y^{(6)}(0) = -1, \quad y^{(7)}(0) = 0, \quad y^{(8)}(0) = 1, \quad y^{(9)}(0) = 0.$
Therefore, the solution of (20) is

$$y(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

which is the closed form solution of the exact solution for the equation (20).

Problem 3.5
Consider the nonlinear Volterra integral equation

$$y(x) = \cos(x) + \frac{1}{2}\sin(2x) + 3x - 2\int_0^x \big(1 + y^2(t)\big)dt,\ 0 < x < 1. \tag{21}$$

The exact solution is $y(x) = \cos x$.
Source: Odibat [11].
Here, $f(x) = \cos(x) + \frac{1}{2}\sin(2x) + 3x$ and $k(x,t) = g(x)h(t)$, where $g(x) = -2$, $h(t) = \frac{1}{y(t)} + y(t)$.
Applying Theorem 2.1, we obtain:
$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0, \quad y'^v(0) = 1,$
$y^{(5)}(0) = 0, \quad y^{(6)}(0) = -1, \quad y^{(7)}(0) = 0, \quad y^{(8)}(0) = 1, \quad y^{(9)}(0) = 0.$

Consequently, the solution of (21) is

$$y(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

which is the closed form solution of the exact solution for the equation (21).

4. CONCLUSION

In this paper, a simple but efficient Taylor series method applicable for the solution of both linear and nonlinear Volterra integral equations of second kind with separable kernels is presented. The method is devoid of complexities associated with some other conventional methods for obtaining the solution of Volterra integral equations and the solution is presented in closed form. A form of Pascal triangle is developed for deriving derivatives of the function $y(x)$. The method does not require linearization, perturbation or discretization for it to be applied on nonlinear problems. Some test problems in the literature were solved to illustrate the performance of the proposed new methods.

REFERENCES

[1] Adomian, G.(1994), Solving Frontier Problems of Physics: The decomposition method, Kluwer.

[2] Barikbin, M. S., Vahidi, A. R. and Damercheli, T. (2021), Solving Volterra Integral Equations of the Second Kind with Convolution Kernel, *Int. J. Industrial Mathematics*, **13**, 7 pages.

[3] Brunner, H. (1992), Implicitly linear collocation methods for nonlinear Volterra equations, *Applied Numerical Mathematics*, **9**, 235–247 .

[4] Costarelli, D. and Spigler, R. (2013), Solving Volterra Integral Equations of the Second Kind by Sigmoidal functions approximation, *Journal of Integral Equations and Applications*, **25**, 193–205.

[5] Huabsomboon, P., Novaprateep, B. and Kaneko, H. (2010), On Taylor-series expansion methods for the second kind integral equations, *Journal of Computational and Applied Mathematics*, **234**, 1466–1472.

[6] Huabsomboon, P., Novaprateep, B. and Kaneko, H. (2010), Taylor-series expansion method for volterra integral equations of the second kind, *Scientiae Mathematicae Japonicae Online*, **e-2010**, 541–551.

[7] Lichae, B. H., Biazar J. and Ayati. Z (2018), A class of Runge–Kutta methods for nonlinear Volterra integral equations of the second kind with singular kernels, *Advances in Difference Equations*, **2018**, Article number: 349.

[8] Maleknejad, K., and Aghazadeh, N. (2005), Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method, *Appl. Math. Comput.*, **161**, 915–922.

[9] Maleknejad K., Aghazadeh N. and Rabbani M. (2006), Numerical computational solution of second kind Fredholm integral equations system by using a Taylor-series expansion method, *Appl. Math. Comput.*, **175**, 1229–1234.

[10] Maleknejad, K., and Damercheli, T. (2014), Improving the accuracy of solutions of the linear second kind Volterra integral equations system by using the Taylor expansion method, *Indian J. Pure Appl. Math*, **45**, 363–376.

[11] Odibat Zaid M. (2008), Differential transform method for solving Volterra integral equation with separable kernels, *Mathematical and Computer Modelling*, **48**, 1144–1149.

[12] Ogunlaran, O.M. and Akinlotan, O.F. (2013), Computational Method for System of Linear Fredholm Integral Equations, *Mathematical Theory and Modeling*, **3**, 1–7.

[13] Ogunlaran, O.M. and Oke, M.O (2013), A Numerical Approach for Solving First Order Integro-Differential Equations, *American Journal of Computational and Applied Mathematics*, **3**, 214–219.

[14] Ren, Y., Zhang, B. and Qiao, H. (1999), A simple Taylor-series expansion method for a class of second integral equations, *J. Comput. Appl. Math.*, **110**, 15–24.

[15] Rocha, A. M.,Azevedo,J. S., Oliveira S. P., Correa, M.R. (2018), Numerical analysis of a collocation method for functional integral equations, Applied Numerical Mathematics **134**, 31–45.

[16] Diogo, T., McKee, S. and Tang, T. (1992), A Hermite-type collocation method for the solution of an integral equation with a certain weakly singular kernel, *IMA Journal of Numerical Analysis*, **11**, 595–605.

[17] Tang, T., McKee, S. and Diogo, T. (1992), Product integration methods for an integral equation with logarithmic singular kernel, *Applied Numerical Mathematics*, **9**, 259–266.

[18] Wazwaz (2011), A.M. Linear and Nonlinear Integral Equations: Methods and Applications, Higher Education, Beijing.

[19] Wazwaz, A.M.(1997), A First Course in Integral Equations, World Scientific, Singapore.

[20] Wazwaz, A.M.(1999), A reliable modification of the Adomian decomposition method, *Appl. Math. Comput.*, **102**, 77–86.

[21] Wazwaz, A.M. and El-Sayed, S. M. (2001), A new modification of the Adomian decomposition method for linear and nonlinear operators, *Applied Mathematics and Computation*, **122**, 393–405. .