Numerical Solutions of Wu-Zhang Equations

Razaq Adekola Oderinu, Johnson Adekunle Owolabi, *Member, IAENG,* and Olufemi Elijah Ige

***Abstract*—In this paper, the Kamal decomposition transform method is used to obtain the approximate numerical solutions of the Wu-Zhang equation. The nonlinear terms were decomposed using the Adomian polynomial, and the results obtained were compared with the results by the modified Adomian decomposition method (MADM) and modified variation iteration method (MVIM). The comparisons revealed that the Kamal decomposition transform method is more effective and accurate for solving any system of nonlinear partial differential equations.**

***Index Terms*—Kamal transform, Adomian polynomial, Wu-Zhang equation.**

I. Introduction

Wu and Zhang derived three sets of model equations from modeling nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth [1]. One of these Wu-Zhang equations describes $(2+1)$-dimensional dispersive long wave, and the other nonlinear equations are essential for the coastal and civil engineers to model in the harbor and coastal design [2]. The equations are given as follows:

$$u_t + uu_x + vu_y + w_x = 0, \tag{1}$$

$$v_t + uv_x + vv_y + w_y = 0, \tag{2}$$

$$w_t+(uw)_x+(vw)_y+\frac{1}{3}(u_{xxx}+v_{yyy}+u_{xyy}+v_{xxy}) = 0. \tag{3}$$

where u is the surface velocity of water along the x-direction, v is the surface velocity of the water along the y-direction, and w is the elevation of the water.

Several researchers have used different methods to obtain the approximate analytical solutions of these Wu-Zhang equations. Some of these methods include variational iteration method [4], modified Adomian decomposition method [3], homotopy perturbation method [5], reduced differential transform method [2], modified variational iteration method [6], and several others.

In this paper, the numerical solutions of Wu-Zhang equations were obtained by Kamal transform and Adomian polynomial. The equations were solved, and the results were compared with the existing solutions from other methods. The errors were calculated to show the accuracy and efficacy of the method.

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II. Properties of Kamal transform

The Kamal transform is defined for the function of exponential order [11]. Consider the functions in the set S defined by

$$S = \left\{ f(t) : k_1, k_2 > 0, |f(t)| < Me^{|t|/k_j}, t \in (-1)^j \times [0, \infty) \right\},$$

where M is a constant which must be a finite number, k_1, k_2 are also constants that can be finite or infinite. Kamal transform is denoted by the operator $K(\cdot)$ which is defined as

$$K\left[f(t)\right] = \int_0^\infty f(t)e^{-t/v}dt = G(v), \quad t \geq 0, \quad k_1 \leq v \leq k_2. \tag{4}$$

Theorem

Let $G(v)$ be a Kamal transform of $f(t)$, where $K[f(t)] = G(v)$, then

1. $K[f'(t)] = \frac{1}{v}G(v) - f(0)$,
2. $K[f''(t)] = \frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0)$,
3. $K[f^n(t)] = \frac{1}{v^n}G(v) - \sum_{k=0}^{n-1} v^{k-n+1}f^k(0)$.

Proof: Applying equation (4) on the first theorem, and using integrating by parts gives

$$K\left[f'(t)\right] = \int_0^\infty f'(t)e^{-t/v}dt,$$
$$= -f(0) + \frac{1}{v}\int_0^\infty f(t)e^{-t/v}dt.$$

Therefore,

$$K\left[f'(t)\right] = \frac{1}{v}G(v) - f(0). \tag{5}$$

Similarly, applying equation (4) on the second theorem, and using integrating by part gives

$$K\left[f''(t)\right] = \int_0^\infty f''(t)e^{-t/v}dt,$$
$$= -f'(0) + \frac{1}{v}\int_0^\infty f'(t)e^{-t/v}dt,$$

then,

$$K\left[f''(t)\right] = -f'(0) + \frac{1}{v}\left(\frac{1}{v}G(v) - f(0)\right).$$

Therefore,

$$K\left[f''(t)\right] = \frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0). \tag{6}$$

The last theorem is proved by using mathematical induction. If $n = 1$ in the third theorem, then equation (5) is gotten again, and it shows that $n = 1$ is true.

Let $n = N$, and assume that it is true so that the equation becomes

$$K[f^N(t)] = \frac{1}{v^N}G(v) - \sum_{k=0}^{N-1} v^{k-N+1}f^k(0).$$

Suppose that $n = N$ is true, then we must prove that $n = N+1$ is also true so that

$$K[f^{N+1}(t)] = \frac{1}{v^{N+1}}G(v) - \sum_{k=0}^{N} v^{k-N} f^k(0).$$

For

$$K[f^{N+1}(t)] = K\left[(fN(t))'\right], \tag{7}$$

then by simplification, equation (7) gives

$$K[f^{N+1}(t)] = \frac{1}{v}K[f^N(t)] - vf^k(0).$$

Applying the expression established earlier, we have

$$K[f^{N+1}(t)] = \frac{1}{v^{N+1}}G(v) - \sum_{k=0}^{N} v^{k-N+1} f^k(0) - vf^k(0).$$

This is expressed as

$$K[f^{N+1}(t)] = \frac{1}{v^{N+1}}G(v) - \sum_{k=0}^{N} v^{k-N} f^k(0).$$

Hence, $n = N+1$ is true, and so

$$K[f^n(t)] = \frac{1}{v^n}G(v) - \sum_{k=0}^{n-1} v^{k-n+1} f^k(0). \tag{8}$$

■

III. Methodology

The detailed procedure used in this research work is given in this section. Our focus is to obtain the approximate analytic solutions of the Wu-Zhang equations. For convenience, the Adomian polynomial is incorporated into the scheme of Kamal transform to obtain a new scheme.
According to [7] and [9].

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \tag{9}$$

where L is an invertible operator, R is the linear differential operator, N indicates the nonlinear term of a differential equation, and $f(x,t)$ is the source term.
By introducing the Kamal transform to equation (9), it yields

$$K[Lu(x,t)] + K[Ru(x,t)] + K[Nu(x,t)] = K[f(x,t)], \tag{10}$$

where

$$K[Lu(x,t)] = \frac{K[u(x,t)]}{v^n} - \sum_{k=0}^{n-1} v^{k-n+1} f^k(0). \tag{11}$$

Substituting equation (11) into equation (10) gives

$$\frac{K[u(x,t)]}{v^n} - \sum_{k=0}^{n-1} v^{k-n+1} f^k(0) + K[Ru(x,t)] + K[Nu(x,t)] = K[f(x,t)].$$

Thus,

$$\frac{K[u(x,t)]}{v^n} = K[f(x,t)] + \sum_{k=0}^{n-1} v^{k-n+1} f^k(0) - \{K[Ru(x,t)] + K[Nu(x,t)]\}. \tag{12}$$

Furthermore, simplifying equation (12), and the resulting expression is given as

$$K[u(x,t)] = v^n K[f(x,t)] + \sum_{k=0}^{n-1} v^{k+1} f^k(0) - v^n\{K[Ru(x,t)] + K[Nu(x,t)]\}. \tag{13}$$

Introducing the inverse Kamal transform to equation (13), this gives

$$u(x,t) = K^{-1}\left[v^n K[f(x,t)] + \sum_{k=0}^{n-1} v^{k+1} f^k(0)\right] - K^{-1}\left[v^n\{K[Ru(x,t)] + K[Nu(x,t)]\}\right]. \tag{14}$$

However, for evaluation purposes, equation (14) can be written as

$$u(x,t) = g(x,t) - K^{-1}\left[v^n\{K[Ru(x,t)] + K[Nu(x,t)]\}\right], \tag{15}$$

where $g(x,t)$ denotes the expression that arises from the given initial condition and the source term, and this is given as

$$g(x,t) = K^{-1}\left[\sum_{k=0}^{n-1} v^{k+1} f^k(0) + v^n K[f(x,t)]\right]. \tag{16}$$

The infinite series becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{17}$$

By decomposing the nonlinear term in equation (15), the expression becomes

$$Nu(x,t) = \sum_{m=0}^{\infty} A_m, \tag{18}$$

where A_m is defined as the Adomian polynomial which can be calculated with the aid of the formula given below [8], [10], [12]

$$A_m = \frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \ m = 0,1,\cdots. \tag{19}$$

Simplifying equation (16) yields

$$u_0(x,t) = g(x,t), \tag{20}$$

and the recursive relation is given as

$$u_{n+1} = -K^{-1}\left[v^n\{K[Ru_n(x,t)] + K[A_m]\}\right], \tag{21}$$

where $n = 1,2,3,\cdots$, and $m \geq 0$. The analytical solution $u(x,t)$ can be approximated by a truncated series

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t). \tag{22}$$

IV. Applications

Considering equations (1), (2), and (3) with the initial conditions given below [2], [3], [5], [6].

$$u(x,y,0) = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y), \quad (23)$$

$$v(x,y,0) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y), \quad (24)$$

$$w(x,y,0) = \frac{2}{3}(k_1^2 + k_2^2) sech^2(k_1 x + k_2 y), \quad (25)$$

where b_0, k_1, k_2, and k_3 are constants.
By applying the Kamal transform [11] on equations (1), (2), and (3), the expression is as follows

$$K[u_t] = -K[uu_x + vu_y + w_x], \quad (26)$$

$$K[v_t] = -K[uv_x + vv_y + w_y], \quad (27)$$

$$K[w_t] = -K[(uw)_x + (vw)_y] + K\left[\frac{1}{3}(u_{xxx} + v_{yyy} + u_{xyy} + v_{xxy})\right]. \quad (28)$$

Simplifying equations (26), (27), and (28) gives

$$\frac{u(x,y,t)}{v} - u(x,y,0) = -K[uu_x + vu_y + w_x], \quad (29)$$

$$\frac{v(x,y,t)}{v} - v(x,y,0) = -K[uv_x + vv_y + w_y], \quad (30)$$

$$\frac{w(x,y,t)}{v} - w(x,y,0) = -K[(uw)_x + (vw)_y] + K\left[\frac{1}{3}(u_{xxx} + v_{yyy} + u_{xyy} + v_{xxy})\right]. \quad (31)$$

Applying the initial conditions given in equations (23), (24), and (25) in equations (29), (30), and (31) yields

$$u(x,y,t) = v\left[-\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y)\right] - vK[uu_x + vu_y + w_x], \quad (32)$$

$$v(x,y,t) = v\left[b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y)\right] - vK[uv_x + vv_y + w_y], \quad (33)$$

$$w(x,y,t) = v\left[\frac{2}{3}(k_1^2 + k_2^2) sech^2(k_1 x + k_2 y)\right] + vK\left[(uw)_x + (vw)_y + \frac{1}{3}(u_{xxx} + v_{yyy} + u_{xyy} + v_{xxy})\right]. \quad (34)$$

Applying the Kamal inverse transform on equations (32), (33), and (34) to gives

$$u_0(x,y,t) = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y),$$

$$v_0(x,y,t) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y),$$

$$w_0(x,y,t) = \frac{2}{3}(k_1^2 + k_2^2) sech^2(k_1 x + k_2 y).$$

And thus, the recursive relation are as follows

$$u_{n+1} = -K^{-1}\left[v\left\{K\left[\sum_{k=0}^{n-1} A_n + \sum_{k=0}^{n-1} B_n + \left(\sum_{k=0}^{\infty} w_n\right)_x\right]\right\}\right], \quad (35)$$

$$v_{n+1} = -K^{-1}\left[v\left\{K\left[\sum_{k=0}^{\infty} C_n + \sum_{k=0}^{\infty} D_n + \left(\sum_{k=0}^{\infty} w_n\right)_y\right]\right\}\right], \quad (36)$$

$$w_{n+1} = -K^{-1}\left[vK\left[\sum_{k=0}^{\infty} E_n + \sum_{k=0}^{\infty} F_n\right]\right] + K^{-1}\left[vK\left[\frac{1}{3}\left(\left(\sum_{k=0}^{\infty} u_n\right)_{xxx} + \left(\sum_{k=0}^{\infty} v_n\right)_{yyy}\right)\right]\right] + K^{-1}\left[vK\left[\frac{1}{3}\left(\left(\sum_{k=0}^{\infty} u_n\right)_{xyy} + \left(\sum_{k=0}^{\infty} v_n\right)_{xxy}\right)\right]\right], \quad (37)$$

where A_n, B_n, C_n, D_n, E_n, and F_n are the nonlinear terms which are decomposed by using Adomian polynomial as follows

$A_0 = u_0(u_0)_x,$
$A_1 = u_1(u_0)_x + u_0(u_1)_x,$
$A_2 = u_2(u_0)_x + u_1(u_1)_x + u_0(u_2)_x,$
$A_3 = u_3(u_0)_x + u_2(u_1)_x + u_1(u_2)_x + u_0(u_3)_x,$
$A_4 = u_4(u_0)_x + u_3(u_1)_x + u_2(u_2)_x + u_1(u_3)_x + u_0(u_4)_x,$

$B_0 = v_0(u_0)_y,$
$B_1 = v_1(u_0)_y + v_0(u_1)_y,$
$B_2 = v_2(u_0)_y + v_1(u_1)_y + v_0(u_2)_y,$
$B_3 = v_3(u_0)_y + v_2(u_1)_y + v_1(u_2)_y + v_0(u_3)_y,$
$B_4 = v_4(u_0)_y + v_3(u_1)_y + v_2(u_2)_y + v_1(u_3)_y + v_0(u_4)_y,$

$C_0 = u_0(v_0)_x,$
$C_1 = u_1(v_0)_x + u_0(v_1)_x,$
$C_2 = u_2(v_0)_x + u_1(v_1)_x + u_0(v_2)_x,$
$C_3 = u_3(v_0)_x + u_2(v_1)_x + u_1(v_2)_x + u_0(v_3)_x,$
$C_4 = u_4(v_0)_x + u_3(v_1)_x + u_2(v_2)_x + u_1(v_3)_x + u_0(v_4)_x,$

$D_0 = v_0(v_0)_y,$
$D_1 = v_1(v_0)_y + v_0(v_1)_y,$
$D_2 = v_2(v_0)_y + v_1(v_1)_y + v_0(v_2)_y,$
$D_3 = v_3(v_0)_y + v_2(v_1)_y + v_1(v_2)_y + v_0(v_3)_y,$
$D_4 = v_4(v_0)_y + v_3(v_1)_y + v_2(v_2)_y + v_1(v_3)_y + v_0(v_4)_y,$

$E_0 = (u_0 w_0)_x,$
$E_1 = (u_0 w_1 + u_1 w_0)_x,$
$E_2 = (u_0 w_2 + u_1 w_1 + u_2 w_0)_x,$
$E_3 = (u_0 w_3 + u_1 w_2 + u_2 w_1 + u_3 w_0)_x,$
$E_4 = (u_0 w_4 + u_1 w_3 + u_2 w_2 + u_3 w_1 + u_4 w_0)_x,$

$F_0 = (v_0 w_0)_y,$
$F_1 = (v_0 w_1 + v_1 w_0)_y,$
$F_2 = (v_0 w_2 + v_1 w_1 + v_2 w_0)_y,$
$F_3 = (v_0 w_3 + v_1 w_2 + v_2 w_1 + v_3 w_0)_y,$
$F_4 = (v_0 w_4 + v_1 w_3 + v_2 w_2 + v_3 w_1 + v_4 w_0)_y.$

Substituting the above-mentioned Adomian polynomial into equations (35), (36), and (37), then simplifying. The solutions obtained were as follows

$$u_0 = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y),$$

$$u_1 = \frac{2}{3} \frac{k_1 k_3 t \sqrt{3}}{\cosh(xk_1 + yk_2)^2},$$

$$u_2 = -\frac{2}{3} \frac{t^2 k_1 k_3^2 \sqrt{3} \sinh(xk_1 + yk_2)}{\cosh(xk_1 + yk_2)^3},$$

$$u_3 = \frac{2}{9} \frac{t^3 k_1 k_3^3 \sqrt{3}(2\cosh(xk_1 + yk_2)^2 - 3)}{\cosh(xk_1 + yk_2)^4},$$

$$u_4 = -\frac{2}{9} \frac{t^4 k_1 k_3^4 \sqrt{3} \sinh(xk_1 + yk_2)(\cosh(xk_1 + yk_2)^2 - 3)}{\cosh(xk_1 + yk_2)^5},$$

$$u_5 = \frac{2}{45} \frac{t^5 k_1 k_3^5 \sqrt{3}(2\cosh(xk_1 + yk_2)^4 - 15\cosh(xk_1 + yk_2)^2 + 15)}{\cosh(xk_1 + yk_2)^6}.$$

Also,

$$v_0 = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y),$$

$$v_1 = \frac{2}{3} \frac{k_2 k_3 t \sqrt{3}}{\cosh(xk_1 + yk_2)^2},$$

$$v_2 = -\frac{2}{3} \frac{t^2 k_2 k_3^2 \sqrt{3} \sinh(xk_1 + yk_2)}{\cosh(xk_1 + yk_2)^3},$$

$$v_3 = \frac{2}{9} \frac{t^3 k_2 k_3^3 \sqrt{3}(2\cosh(xk_1 + yk_2)^2 - 3)}{\cosh(xk_1 + yk_2)^4},$$

$$v_4 = -\frac{2}{9} \frac{t^4 k_2 k_3^4 \sqrt{3} \sinh(xk_1 + yk_2)(\cosh(xk_1 + yk_2)^2 - 3)}{\cosh(xk_1 + yk_2)^5},$$

$$v_5 = \frac{2}{45} \frac{t^5 k_2 k_3^5 \sqrt{3}(2\cosh(xk_1 + yk_2)^4 - 15\cosh(xk_1 + yk_2)^2 + 15)}{\cosh(xk_1 + yk_2)^6}.$$

And,

$$w_0 = \frac{2}{3}(k_1^2 + k_2^2) sech^2(k_1 x + k_2 y),$$

$$w_1 = -\frac{4}{3} \frac{t k_3 \sinh(k_1 x + k_2 y)(k_1^2 + k_2^2)}{\cosh(k_1 x + k_2 y)^3},$$

$$w_2 = -\frac{2}{3} \frac{t^2 k_3^2 (2k_1^2 \cosh(k_1 x + k_2 y)^2 + 2k_2^2 \cosh(k_1 x + k_2 y)^2 - 3(k_1^2 + k_2^2))}{\cosh(k_1 x + k_2 y)^4},$$

$$w_3 = -\frac{8}{9} \frac{t^3 k_3^3 \sinh(k_1 x + k_2 y)(k_1^2 \cosh(k_1 x + k_2 y)^2 + k_2^2 \cosh(k_1 x + k_2 y)^2 - 3(k_1^2 + k_2^2))}{\cosh(k_1 x + k_2 y)^5},$$

$w_4 = \frac{2}{9} \frac{1}{\cosh(k_1 x + k_2 y)^6} \; (t^4 k_3^4 (2\cosh(k_1 x + k_2 y)^4 \; (k_1^2 + 2\cosh(k_1 x + k_2 y)^4 k_2^2 - 15k_1^2 \cosh(k_1 x + k_2 y)^2 - 15k_2^2 \cosh(k_1 x + k_2 y)^2 + 15k_1^2 + 15k_2^2)),$

$w_5 = -\frac{4}{45} \frac{1}{\cosh(k_1 x + k_2 y)^7} (t^5 k_3^5 \sinh(k_1 x + k_2 y)(2\cosh(k_1 x + k_2 y)^4 k_1^2 + 2\cosh(k_1 x + k_2 y)^4 k_2^2 - 30k_1^2 \cosh(k_1 x + k_2 y)^2 - 30k_2^2 \cosh(k_1 x + k_2 y)^2 + 45k_1^2 + 45k_2^2)).$

The approximate solutions of the system of equations (1), (2), and (3) are obtained as:

$$u(x,y,t) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots ,$$

$$v(x,y,t) = v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + \cdots ,$$

$$w(x,y,t) = w_0 + w_1 + w_2 + w_3 + w_4 + w_5 + \cdots .$$

The solutions converge to the exact solutions in [5] as:

$$u(x,y,t) = -\frac{k_3 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y + k_3 t),$$

$$v(x,y,t) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y + k_3 t),$$

$$w(x,y,t) = \frac{2}{3}(k_1^2 + k_2^2) sech^2(k_1 x + k_2 y + k_3 t).$$

Therefore, the figures of the existing exact solutions and the approximate analytical solutions obtained using the Kamal transform method are presented. In addition, the comparison of the absolute errors with other known methods is presented in tabular form, as seen below.

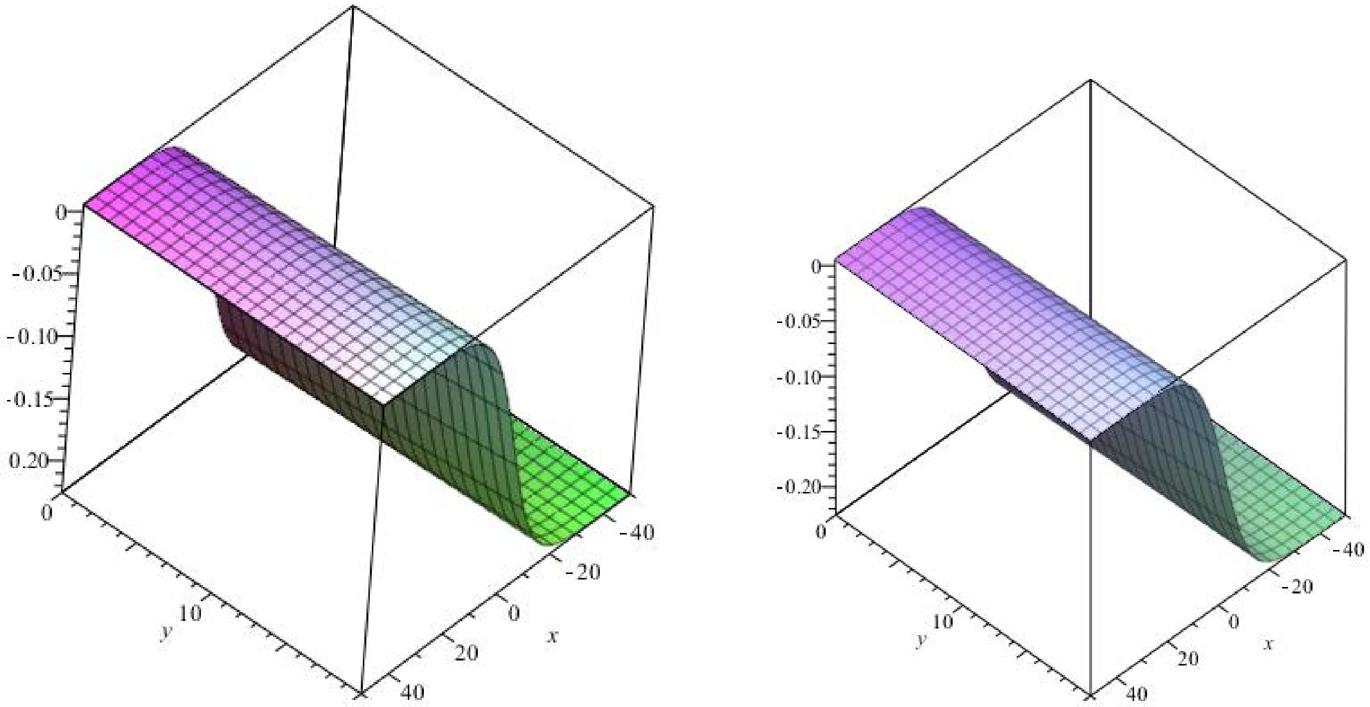

Fig. 1: The left panel shows the exact solution $u(x,y,t)$ of the equation (1)-(3) under the initial conditions (23) - (25) when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, and $t = 5$, while the numerical solution is shown on the right panel under the same initial conditions.

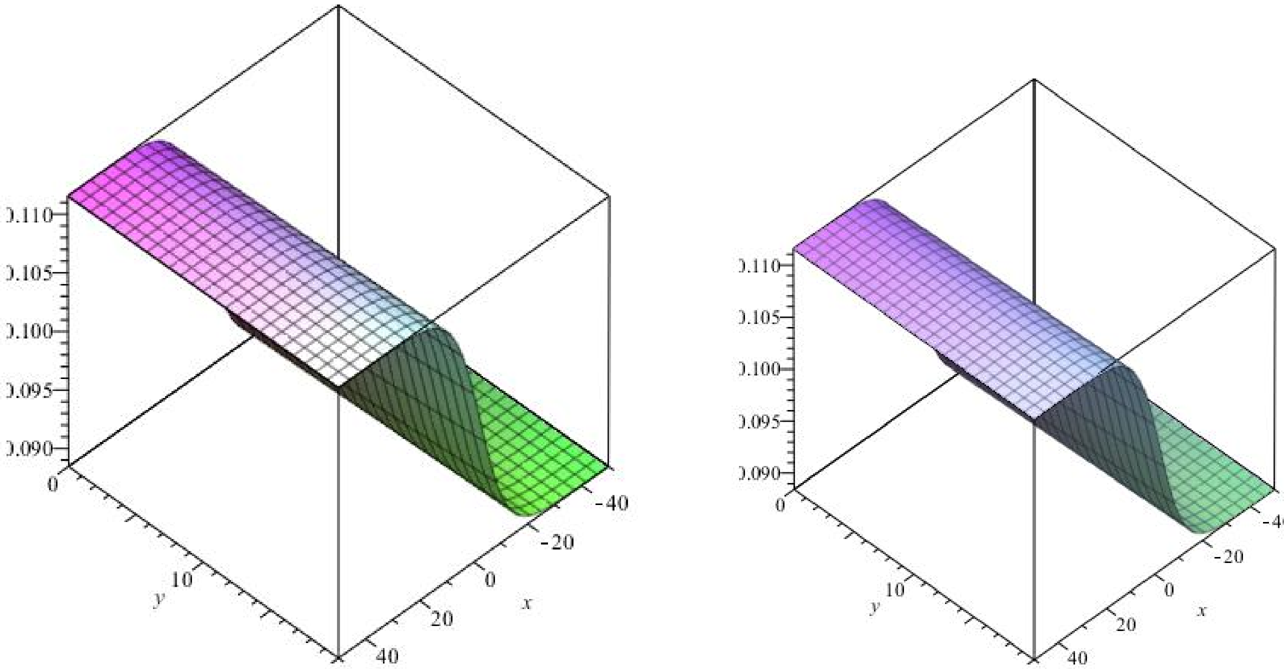

Fig. 2: The left panel shows the exact solution $v(x, y, t)$ of the equation (1) - (3) under the initial conditions (23) - (25) when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, and $t = 5$, while the numerical solution is shown on the right panel under the same initial conditions.

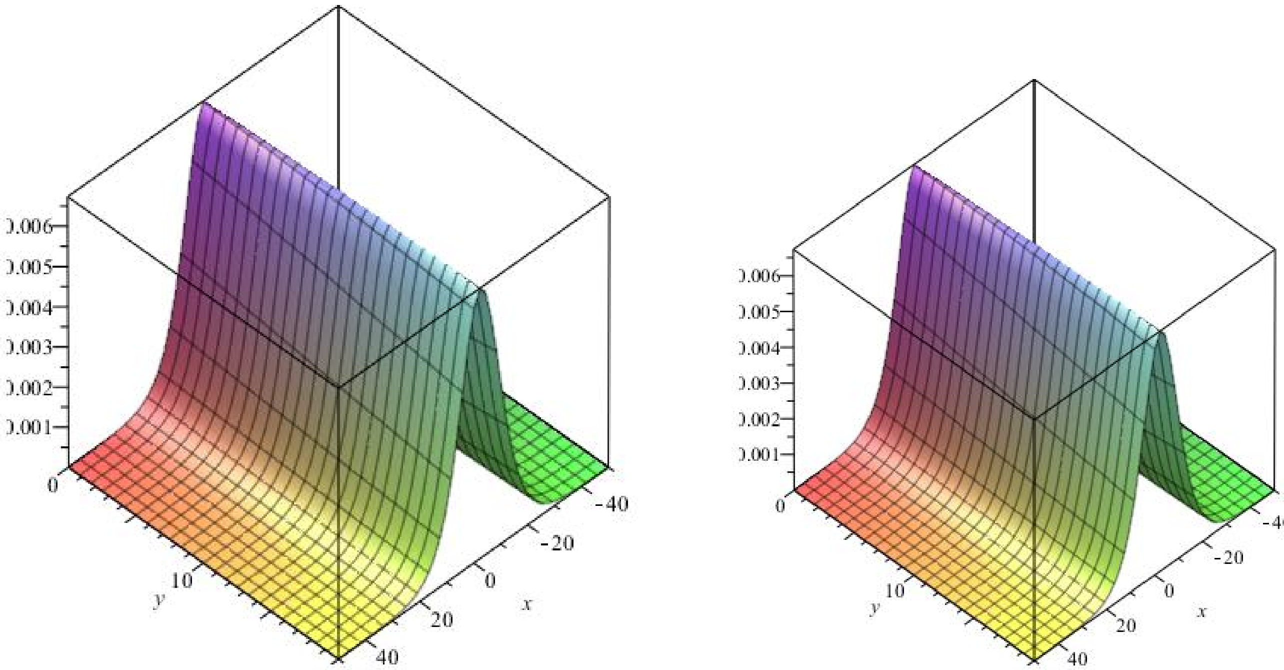

Fig. 3: The left panel shows the exact solution $w(x, y, t)$ of equations (1) - (3) under the initial conditions (23) - (25) when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, and $t = 5$, while the numerical solution is shown on the right panel under the same initial conditions.

TABLE I: Comparison of absolute errors for values of $u(x,t)$ for the first four iterations in Wu-Zhang system between Kamal decomposition transform method (KDTM), modified variation iteration method (MVIM) [6], and modified Adomian decomposition method (MADM) [3] when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, $t = 5$, and $y = 20$.

x	u_{exact}	$\lvert u_{exact} - u_{MVIM}\rvert$ [6]	$\lvert u_{exact} - u_{MADM}\rvert$[3]	$\lvert u_{exact} - u_{KDTM}\rvert$
-50	-0.22545277668	1.6451825E-06	10.39242830E-10	1.317571408E-12
-40	-0.22534239508	1.2148126E-05	4.330797530E-10	9.6316072880E-12
-30	-0.22453009579	8.9315718E-05	3.594419403E-09	6.389835542E-11
-20	-0.21870068411	6.3605735E-04	2.448538578E-08	1.487347756E-10
-10	-0.18334068373	3.6024380E-03	1.023749896E-07	2.292092484E-09
0	-0.081719228866	6.4447985E-03	3.048061683E-07	3.2111582694E-09
10	-0.012048642424	1.9107842E-04	8.922341521E-03	8.3581755996E-10
20	0.002932728123	2.9196656E-04	5.192973059E-09	1.50410411982E-10
30	0.005123370645	4.0172038E-05	1.372972068E-09	2.9830061166E-11
40	0.005423074389	5.44887493E-06	2.078596792E-10	4.22761272016E-12
50	0.005463694742	7.37648418E-07	5.196177359E-11	5.79068686671E-13

TABLE II: Comparison of absolute errors for values of $u(x,t)$ for the first five iterations in Wu-Zhang system between Kamal decomposition transform method (KDTM), modified variation iteration method (MVIM) [6], and modified Adomian decomposition method (MADM) [3] when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, $t = 5$, and $y = 20$.

x	u_{exact}	$\lvert u_{exact} - u_{MVIM}\rvert$ [6]	$\lvert u_{exact} - u_{MADM}\rvert$[3]	$\lvert u_{exact} - u_{KDTM}\rvert$
-50	-0.22545276882	1.6451825E-06	10.39242830E-10	1.6954785693E-14
-40	-0.225342395080	1.2148126E-05	4.330797530E-10	1.5406825142E-13
-30	-0.224530095797	8.9315718E-05	3.594419403E-09	9.2122894666E-13
-20	-0.218700684119	6.3605735E-04	2.448538578E-08	2.6046523737E-12
-10	-0.183340683727	3.6024380E-03	1.023749896E-07	8.9106575340E-12
0	-0.081719228866	6.4447985E-03	3.048061683E-07	1.14315145811E-10
10	-0.0120486424242	1.9107842E-04	8.922341521E-03	2.98347114032E-11
20	0.0029327282125	2.9196656E-04	5.192973059E-09	1.26253690657E-12
30	0.00512337064659	4.0172038E-05	1.372972068E-09	4.67731173117E-13
40	0.0054230743897	5.44887493E-06	2.078596792E-10	6.88739183700E-14
50	0.0054636947425	7.37648418E-07	5.196177359E-11	6.03450181181E-15

TABLE III: Comparison of absolute errors for values of $v(x,t)$ for the first four iterations in Wu-Zhang system between Kamal decomposition transform method (KDTM), modified variation iteration method (MVIM) [6], and modified Adomian decomposition method (MADM) [3] when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, $t = 5$, and $y = 20$.

x	v_{exact}	$\lvert v_{exact} - v_{MVIM}\rvert$[6]	$\lvert v_{exact} - v_{MADM}\rvert$[3]	$\lvert v_{exact} - v_{KDTM}\rvert$
-50	0.088454723115	8.08571164E-09	1.039242830E-11	1.31757140837E-13
-40	0.088465764910	5.96388798E-08	6.330797530E-11	9.63160728803E-13
-30	0.088546990420	4.34887678E-07	3.394419403E-10	6.3898355426E-12
-20	0.089129931588	2.91626141E-06	2.438538578E-09	1.4873477566E-11
-10	0.0926659316272	1.07971370E-05	1.027749896E-08	2.2920924842E-10
0	0.1028280877113	5.875I9681E-06	3.050061683E-08	3.2111582693E-10
10	0.109795135757	7.18185473E-06	8.962341521E-09	8.3581755996E-11
20	0.111293272812	1.30807366E-06	5.392973059E-10	1.504104119828E-11
30	0.1115123370465	1.84431598E-07	1.072972068E-10	2.9830061166E-12
40	0.111542307438	2.50992772E-08	2.078596792E-11	4.2276127016E-13
50	0.1115463696923	3.39937593E-09	5.196177359E-12	5.7906868667E-14

TABLE IV: Comparison of absolute errors for values of $v(x,t)$ for the first five iterations in Wu-Zhang system between Kamal decomposition transform method (KDTM), modified variation iteration method (MVIM) [6], and modified Adomian decomposition method (MADM) [3] when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, $t = 5$, and $y = 20$.

x	v_{exact}	$\|v_{exact} - v_{MVIM}\|$[6]	$\|v_{exact} - v_{MADM}\|$[3]	$\|v_{exact} - v_{KDTM}\|$
-50	0.0884547231157	8.08571164E-09	1.039242830E-11	1.69547856936E-15
-40	0.0884657604920	5.96388798E-08	6.330797530E-11	1.54068251420E-14
-30	0.0885469904204	4.34887678E-07	3.394419403E-10	9.21228946667E-14
-20	0.0891299315299	2.91626141E-06	2.438538578E-09	2.60465237374E-13
-10	0.0926659316281	1.07971370E-05	1.027749896E-08	8.9106575340E-13
0	0.1028280771133	5.875I9681E-06	3.050061683E-08	1.143151458119E-11
10	0.1097951357575	7.18185473E-06	8.962341521E-09	2.98347114032E-12
20	0.1112932728123	1.30807366E-06	5.392973059E-10	1.26253690657E-13
30	0.1115123370645	1.84431598E-07	1.072972068E-10	4.67731173117E-14
40	0.1115423074389	2.50992772E-08	2.078596792E-11	6.88739183700E-15
50	0.1115463694742	3.39937593E-09	5.196177359E-12	6.03450181181E-16

TABLE V: Comparison of absolute errors for values of $w(x,t)$ for the first four iterations in Wu-Zhang system between Kamal decomposition transform method (KDTM), modified variation iteration method (MVIM) [6], and modified Adomian decomposition method (MADM) [3] when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, $t = 5$, and $y = 20$.

x	w_{exact}	$\|w_{exact} - w_{MVIM}\|$[6]	$\|w_{exact} - w_{MADM}\|$ [3]	$\|w_{exact} - w_{KDTM}\|$
-50	2.015707514E-06	3.705249242E-07	7.7498805E-12	1.53902593566E-13
-40	1.487994142E-05	2.73I8146111E-06	5.708491389E-11	1.105232099077E-12
-30	1.091761976E-04	1.986047907E-05	4.118114838E-10	6.49361688562E-12
-20	7.663347764E-04	1.302774181E-04	2.634558628E-09	1.83876153386E-11
-10	4.0170111090E-03	1.07971370E-05	1.027749896E-08	6.0411550366E-11
0	6.329433315E-03	1.153652836E-04	3.839847338E-08	8.02431620060E-10
10	1.8881267656E-03	2.265746793E-05	1.484359887E-08	2.08314865983E-10
20	2.9266378718E-04	6.972254850E-07	5.892963187E-11	8.84847649359E-12
30	4.0371146550E-05	1.991078199E-07	1.529342660E-10	3.29135644420E-12
40	5.4778520393E-06	2.87710754E-08	2.418224902E-11	4.89398587782E-13
50	7.41607526114E-07	3.95110044E-09	3.337771675E-12	6.7052948299E-14

TABLE VI: Comparison of absolute errors for values of $w(x,t)$ for the first five iterations in Wu-Zhang system between Kamal decomposition transform method (KDTM), modified variation iteration method (MVIM) [6], and modified Adomian decomposition method (MADM) [3] when $b_0 = k_1 = 0.1$, $k_2 = k_3 = 0.01$, $t = 5$, and $y = 20$.

x	w_{exact}	$\|w_{exact} - w_{MVIM}\|$[6]	$\|w_{exact} - w_{MADM}\|$ [3]	$\|w_{exact} - w_{KDTM}\|$
-50	2.01570751488E-06	3.705249242E-07	7.7498805E-12	2.5476222459E-15
-40	1.487994142427E-05	2.73I8146111E-06	5.708491389E-11	1.7768141946E-14
-30	1.09176197648E-04	1.986047907E-05	4.118114838E-10	7.7149545148E-14
-20	7.66334776494E-04	1.302774181E-04	2.634558628E-09	1.2180896703E-12
-10	4.01701110909E-03	1.07971370E-05	1.027749896E-08	1.324799745967E-11
0	6.329433315296E-03	1.153652836E-04	3.839847338E-08	1.84912607221E-11
10	1.888126765615E-03	2.265746793E-05	1.484359887E-08	3.13172676390E-12
20	2.926637871853E-04	6.972254850E-07	5.892963187E-11	1.161243598747E-13
30	4.037114655079E-05	1.991078199E-07	1.529342660E-10	4.87828233886E-14
40	5.477852039333E-06	2.87710754E-08	2.418224902E-11	8.05878324962E-15
50	7.41607526115E-07	3.95110044E-09	3.337771675E-12	1.11808093287E-15

V. Discussion of Results

The Kamal transform and Adomian polynomial scheme was employed to solve the $(2+1)$-dimensional dispersive long wave Wu-Zhang equation. The iterative solutions obtained were in a series form whose closed-form converges to the exact solution.

The right panels of figures 1 to 3 show the shape of the solutions to the problem considered using KDTM as they competed favorably with the shape of the exact solutions shown on the left panels. The features and physical formation of the equation could also be observed from the shape of figures, as this will enable the engineers and scientists in modeling and analyzing the behavior of such models for prediction purposes.

In addition, Tables I to VI show the errors obtained when the approximate solutions obtained by KDTM and the other methods mentioned earlier in this work were compared with the exact solutions. The errors for the fourth and fifth iterations were calculated for comparison purposes. Tables I, III, and V show the errors of the fourth iterations, while tables II, IV, and VI show the errors of the fifth iterations. The maximum errors obtained from tables I, III, and V are: 3.211×10^{-9}, 3.211×10^{-10}, and 6.04×10^{-11} while the minimum errors obtained from the same tables are: 5.79×10^{-13}, 5.79×10^{-14}, and 6.70×10^{-14} where four iterations of KDTM is applied. Furthermore, the maximum errors obtained for five iterations of KDTM in tables II, IV, and VI are: 1.14×10^{-10}, 1.14×10^{-11}, and 1.84×10^{-11} while the calculated minimum errors from the same tables are 6.03×10^{-15}, 6.03×10^{-16}, and 1.11×10^{-15}. These errors obtained by KDTM were found to be the least compared to errors in [6] and [3] where modified variation iteration method and modified Adomian decomposition method were used, respectively. It was also observed that the errors of five iterations were less than that of four iterations, so the higher the number of iterations considered, the smaller the errors and the better the results achieved.

VI. Conclusion

The Kamal decomposition transform method is explored to obtain the approximate analytical solutions of the Wu-Zhang system of equations. The Adomian polynomial is integrated into the scheme of Kamal transform to decompose the nonlinear terms purposely to obtain the solution of the mentioned problem. The KDTM is used directly and efficiently, with fast convergence that requires fewer computations.

Figures 1 to 3 show rapid convergent approximation that yields the closed-form solution. Furthermore, the comparison with other methods was shown in Table I to III, and the errors obtained show that KDTM is reliable, effective, and accurate. Therefore, KDTM has more advantages, and it can be seen as a powerful mathematical tool for solving any system of nonlinear partial differential equations arising in physics and engineering.

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