Approximate Solution Method for Third-Order Multi-Point Boundary Value Problems

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ABSTRACT

In this paper, we formulate a method using Chebyshev polynomials for solving both linear and non-linear multi-point boundary value problems. Quasi linearization technique is used to transform a non-linear boundary value problem into a sequence of linear boundary value problems with a quadratic polynomial which satisfies the boundary conditions chosen as the initial approximation. Four numerical examples are given to demonstrate the efficiency of the present method and the results obtained are compared with other methods in the literatures.

Keywords- Linear and non-linear problems; Multi-point boundary value problem; Chebyshev polynomials; Quasi linearization technique

1. INTRODUCTION

In recent years, multi-point boundary value problems have received a considerable growing research interest. Multi-point boundary value problems appear in various areas of sciences and engineering. Modelling and analysing problems arising from electric power networks, railway systems, telecommunication lines, construction of large bridges with many supports and analysing kinetic reaction problems are some examples of physical phenomena that lead to multi-point boundary value problems[8, 21]. Many problems in theory of elastic stability can be set up as multi-point boundary value problems [9]. Several numerical methods have been developed for solving multi-point boundary value problems. Some of these are Reproducing kernel method [1,2,5,6,10,11,13], Adomain decomposition method[12], The shooting method[14,15], Weighted residual method[22], Padé approximations[18] and Homotopy perturbation method[23].

Researchers have proposed several methods for handling third-order boundary value problems. For instance, see[4,12,16,17,18] and references therein. However in this paper, we present an approximate method based on Chebyshev polynomials approach for the solution of third-order multi-point boundary value problems of the form

$$a_0(x)y'''(x)+a_1(x)y''(x)+a_2(x)y'(x)+a_3(x)y(x)=f(x) \qquad (1.1)$$

subject to the boundary conditions

$$y(0)=\alpha_1 \qquad (1.2)$$

$$y'(0)=\alpha_2 \qquad (1.3)$$

$$y'(1)=\alpha y'(\eta)+\lambda \qquad (1.4)$$

where $a_0(x),\ a_1(x),\ a_2(x),\ a_3(x)$ *and* $f(x)$ are continuous functions and $\alpha_1,\ \alpha_2,\ \alpha,\ \lambda$ are constants and $\eta\in(0,1)$.

This paper is organised as follows: In the next section, we define Chebyshev and shifted Chebyshev polynomials of the first and second kinds, establish some relationship between the polynomials of the first and second kinds, describe the derivatives of the shifted Chebyshev polynomials and some of its properties. Section 3 summarizes the application of the proposed method to the solution of problem (1.1) – (1.4). Four numerical examples are given in section 4 to demonstrate the applicability and validity of the method. Finally, the concluding remarks are given in section 5.

2. FORMULATION OF THE METHOD

2.1. Definitions of Chebyshev Polynomials

Definition 2.1: The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial of degree n in x defined by

$$T_n(x)=\cos n\theta,\ \text{where } x=\cos\ \theta,\ x\in[-1,1] \qquad (2.1)$$

which satisfies the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n \geq 2 \tag{2.2}$$

with the initial conditions

$$T_0(x) = 1, \quad T_1(x) = x \tag{2.3}$$

Definition 2.2: The shifted Chebyshev polynomial $T_n^*(x)$ of the first kind on [0,1] is a polynomial of degree n in x defined by

$$T_n^*(x) = T_n(2x-1) \tag{2.4}$$

Similarly, $T_n^*(x)$ satisfies the recurrence relation

$$T_n^*(x) = 2(2x-1)T_{n-1}^*(x) - T_{n-2}^*(x),\ n \geq 2 \tag{2.5}$$

with the initial conditions

$$T_0^*(x) = 1, \quad T_1^*(x) = 2x-1 \tag{2.6}$$

Definition 2.3: The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},\ \text{where } x = \cos\ \theta,\ x \in [-1,1] \tag{2.7}$$

The Chebyshev polynomials of the second kind are defined by the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),\ n \geq 2 \tag{2.8}$$

with the initial conditions

$$U_0(x) = 1, \quad U_1(x) = 2x \tag{2.9}$$

Definition 2.4: The shifted Chebyshev polynomial $U_n^*(x)$ of the second kind on [0,1] is a polynomial of degree n in x defined by

$$U_n^*(x) = U_n(2x-1) \tag{2.10}$$

The shifted Chebyshev polynomials of the second kind are defined by recursively by

$$U_n^*(x) = 2(2x-1)U_{n-1}^*(x) - U_{n-2}^*(x),\ n \geq 2 \tag{2.11}$$

with the initial conditions

$$U_0^*(x) = 1, \quad U_1^*(x) = 2(2x-1) \tag{2.12}$$

2.2 Relation between Chebyshev polynomials of the first and second kinds

The following theorems show relationships between the shifted Chebyshev polynomial of the first and second kinds.

Theorem 2.1

Let $T_n^*(x)$ and $U_n^*(x)$ denote the shifted Chebyshev polynomials of degree n in x on [0,1] of the first and second kinds respectively. Then for $n \geq 1$, $\int 2U_{n-1}^*(x)dx = \frac{1}{n}T_n^*(x) + c$, where c is an arbitrary constant.

Proof

The Chebyshev polynomial of the second kind of degree $n-1$ is defined by

$$U_{n-1}(x) = \frac{\sin n\theta}{\sin\theta},\ \text{where } x = \cos\ \theta,\ x \in [-1,1] \tag{2.13}$$

Integrating both sides of (2.13), we obtain

$$\int U_{n-1}(x)\,dx = \frac{\cos n\theta}{n} + c$$

Thus,

$$\int U_{n-1}(x)\,dx = \frac{1}{n}T_n(x) + c \qquad (2.14)$$

We now use the transformation $x = 2t - 1$ to map $x \in [-1,1]$ to $t \in [0,1]$, then (2.14) becomes

$$\int 2U_{n-1}(2t-1)\,dt = \frac{1}{n}T_n(2t-1) + c$$

And since t is a dummy variable, we obtain

$$\int 2U^*_{n-1}(x)\,dx = \frac{1}{n}T^*_n(x) + c \qquad (2.15)$$

Theorem 2.2

Let $T_n^*(x)$ and $U_n^*(x)$ denote the shifted Chebyshev polynomials of degree n in x on [0,1] of the first and second kinds respectively. Then $U^*_{n-1}(x) = 2T^*_{n-1}(x) + U^*_{n-3}(x),\ n \ge 3,$

Proof

From the trigonometry identity

$$\sin(n+1)\theta - \sin(n-1)\theta = 2\cos n\theta \sin n\theta$$

Dividing both sides by $\sin\theta$, we obtain

$$U_n(x) = 2T_n(x) + U_{n-2}(x)$$

Replacing n with $n-1$, we have

$$U_{n-1}(x) = 2T_{n-1}(x) + U_{n-3}(x),\ n \ge 3 \qquad (2.16)$$

Now for $x \in [0,1]$, (2.16) becomes

$$U_{n-1}(2x-1) = 2T_{n-1}(2x-1) + U_{n-3}(2x-1),\ n \ge 3$$

Hence,

$$U^*_{n-1}(x) = 2T^*_{n-1}(x) + U^*_{n-3}(x),\ n \ge 3 \qquad (2.17)$$

Theorem 2.3

For $n \ge 1,\ U^*_{n-1}(x) = 2\sum_{\substack{r=0 \\ (n-r)\text{ odd}}}^{n-1}{}' T^*_r(x)$,

where $T_r^*(x)$ and $U_n^*(x)$ denote the shifted Chebyshev polynomials of degrees r and n in x on [0,1] of the first and second kinds respectively.

Proof

It follows from (2.17) that

$$\left.\begin{array}{l} U_{n-3}^*(x) = 2T_{n-3}^*(x) + U_{n-5}^*(x) \\ U_{n-5}^*(x) = 2T_{n-5}^*(x) + U_{n-7}^*(x) \\ \quad\vdots \\ U_3^*(x) = 2T_3^*(x) + 2T_1^*(x) \text{ for } n=4 \\ \quad \text{or} \\ U_2^*(x) = 2T_2^*(x) + T_0^*(x) \text{ for } n=3 \end{array}\right\} \tag{2.18}$$

Suppose n is odd in (2.17), substituting (2.18) appropriately into (2.17) gives

$$U_{n-1}^*(x) = 2\left(T_{n-1}^*(x) + T_{n-3}^*(x) + T_{n-5}^*(x) + \cdots T_2^*(x) + \tfrac{1}{2}T_0^*(x)\right) \tag{2.19}$$

Hence,

$$U_{n-1}^*(x) = 2\sum_{r=0}^{n-1} T_r^*(x), \tag{2.20}$$

where r is even and a summation symbol with prime denotes a sum with first term halved.

Similarly suppose n is even, we obtain

$$U_{n-1}^*(x) = 2\sum_{r=1}^{n-1} T_r^*(x), \tag{2.21}$$

where r is odd.

Now equations (2.20) and (2.21) can be written as a single equation as

$$U_{n-1}^*(x) = 2 \sum_{\substack{r=0 \\ (n-r) \text{ odd}}}^{n-1}{}' T_r^*(x) \tag{2.22}$$

2.4 Derivatives of Chebyshev Polynomials

Derivatives of the shifted Chebyshev polynomials $T_n^*(x)$ of the first kind in the range [0,1] are obtained as follows:

Differentiating both sides of (2.15) with respect to x, we get

$$\frac{d}{dx} T_n^*(x) = 2nU_{n-1}^*(x) \tag{2.23}$$

Substituting equations (2.22) into (2.23), we get

$$\frac{d}{dx} T_n^*(x) = 4n \sum_{\substack{r=0 \\ (n-r) \text{ odd}}}^{n-1}{}' T_r^*(x) \tag{2.24}$$

Differentiating both sides of (2.24) with respect to x, we obtain after some algebraic manipulations,

$$\frac{d^2}{dx^2} T_n^*(x) = 4\sum_{\substack{r=0 \\ (n-r) \text{ even}}}^{n-2}{}' n(n^2 - r^2) T_r^*(x) \tag{2.25}$$

Likewise for the third derivative, we have

$$\frac{d^3}{dx^3} T_n^*(x) = 16\sum_{l=0}^{n-3}{}' \sum_{\substack{r=l+1 \\ (n-r) \text{ even}}}^{n-2} nr(n^2 - r^2) T_l^*(x) \tag{2.26}$$

2.3 Some Properties of Chebyshev Polynomials

Theorem 2.4

$T_n^*(x)$ of degree $n \geq 1$ assumes its $n+1$ extrema in [0,1] at $x_j = \frac{1}{2}\left(1+\cos\left(\frac{j\pi}{n}\right)\right)$ with $T_n(x_j) = (-1)^j$, for each $j = 0,1, \ldots, n.$

Proof

We use a property of $\cos(x)$ to determine the extrema of the shifted Chebyshev polynomial. Recall that the extrema occur when $\cos(\theta) = \pm 1$ which is a multiple of π. If we then set

$n\cos^{-1}(2x_j - 1) = j\pi$. This implies that

$$x_j = \frac{1}{2}\left(1+\cos\left(\frac{j\pi}{n}\right)\right), j = 0,1, \ldots, n \tag{2.27}$$

Also,

$$\cos\left(n\cos^{-1}(2x_j - 1)\right) = \cos j\pi = (-1)^j$$

Thus,

$$T_n^*(x_j) = (-1)^j, \ j = 0,1, \ldots, n \tag{2.28}$$

From (2.28) we obtain for $j = n$ and $j = 0$, respectively

$$T_n^*(0) = (-1)^n, \ n \geq 1 \tag{2.29}$$

and

$$T_n^*(1) = 1, \ n \geq 1 \tag{2.30}$$

Evaluating (2.24) at the two endpoints, we obtain the following important results:

$$\frac{d}{dx}T_n^*(x)|_{x=0} = (-1)^{n+1}2n^2, \ n \geq 0 \tag{2.12}$$

$$\frac{d}{dx}T_n^*(x)|_{x=1} = 2n^2, n \geq 0 \tag{2.13}$$

3. IMPLEMENTATION OF THE METHOD

In order to solve equation (1.1) with the boundary conditions (1.2) - (1.4), we approximate $y(x)$ using the finite sum

$$y_n(x) = \sum_{k=0}^{n}{}' c_k T_k^*(x) \tag{3.1}$$

where c_k are constants to be determined.

Substituting (3.1) and its derivatives into (1.1), we obtain

$$16a_0(x)\sum_{k=3}^{n}\sum_{\substack{l=0 \\ (k-r)\ even}}^{k-3}{}' \sum_{r=l+1}^{k-2} kr(k^2-r^2)c_k T_l^*(x) + 4a_1(x)\sum_{k=2}^{n}\sum_{\substack{r=0 \\ (k-r)\ even}}^{k-2}{}' k(k^2-r^2)c_k T_r^*(x)$$

$$+4a_2(x)\sum_{k=1}^{n}\sum_{\substack{r=0 \\ (k-r)\ odd}}^{k-1}{}' kc_k T_r^*(x) + a_3(x)\sum_{k=0}^{n}{}' c_k T_r^*(x) = f(x) \tag{3.2}$$

We have to select $n-2$ points in the range of integration such that $y_n(x)$ satisfies the differential equation (1.1) at these $n-2$ collocation points and the boundary conditions (1.2) – (1.4).

Setting $x = x_j$ in (3.2), we obtain

$$16a_0(x_j)\sum_{k=3}^{n}\sum_{\substack{l=0\\(k-r)\ even}}^{k-3}{}' \sum_{r=l+1}^{k-2} kr(k^2-r^2)rc_kT_l^*(x_j)+4a_1(x_j)\sum_{k=2}^{n}\sum_{\substack{r=0\\(k-r)\ even}}^{k-2}{}' k(k^2-r^2)c_kT_r^*(x_j)$$

$$+4a_2(x_j)\sum_{k=1}^{n}\sum_{\substack{r=0\\(k-r)\ odd}}^{k-1}{}' kc_kT_r^*(x_j)+a_3(x_j)\sum_{k=0}^{n}{}' c_kT_r^*(x_j)=f(x_j), \tag{3.3}$$

where x_j are firstly chosen as the evenly spaced points

$$x_j=\tfrac{j}{n-1},\ j=1,2,\ ,\ .\ .\ .,n-2. \tag{3.4}$$

Secondly, to achieve higher accuracy we use unevenly distributed nodes. In this case, the collocation points are chosen to be the internal extrema

$$x_j=\frac{1}{2}\left(1+\cos\left(\tfrac{j\pi}{n-1}\right)\right),\ j=1,2,....,n-2 \tag{3.5}$$

of the $(n-1)th$ order shifted Chebyshev polynomial $T_{n-1}^*(x)$.

Now, the boundary conditions (1.2) – (1.4) yield respectively

$$\sum_{k=0}^{n}{}'(-1)^k c_k=\alpha_1 \tag{3.6}$$

$$\sum_{k=0}^{n}(-1)^{k+1}2k^2c_k=\alpha_2 \tag{3.7}$$

$$\sum_{k=0}^{n}2k^2c_k=4\alpha\sum_{k=1}^{n}\sum_{\substack{r=0\\(k-r)\ odd}}^{k-1}{}' kc_kT_r^*(\eta)+\lambda \tag{3.8}$$

Therefore, the $n-2$ equations obtained in (3.3) plus the 3 boundary conditions (3.6) - (3.8) give the system of (n+1) equations which is solved for the unknown coefficients $c_k,\ k=0,1,...,n$ in (3.1).

4. NUMERICAL EXAMPLES

In this section, two linear and two non-linear examples are examined to test the efficiency of the proposed method. The results obtained show that the method with nonuniformly- distributed nodes gives a better result. All computations are carried out with matlab 2010a.

Problem 4.1

Consider the following linear third-order boundary value problem[19, 1, 3, 7]

$$y'''-xy=(x^3-2x^2-5x-3)e^x \tag{4.1}$$

Subject to the boundary conditions

$$y(0)=0,\ y'(0)=1,\ y'(1)=-e \tag{4.2}$$

The exact solution is $y(x)=x(1-x)e^x$. We report absolute errors of our method for n=8 and n=10 with the two sets of grid points together with the results obtained in [1] at some selected points.

Problem 4.2

Consider the following variable coefficient non-homogeneous linear third-order boundary value problem [1]

$$y'''(x)+xy''(x)=-6x^2+3x-6\quad 0\le x\le 1 \tag{4.3}$$

Subject to the boundary conditions

$$y(0)=0,\ y'(0)=1,\ y'(1)=y'(\tfrac{1}{2})-\tfrac{3}{4} \tag{4.4}$$

The exact solution is $y(x)=\frac{3}{2}x^2-x^3$. The numerical results by our method for n=3 with either way of choosing the grid points (i.e. equation (3.4) or (3.5)) are presented in Table 2. We compare the absolute errors of the proposed method with the method of Akram et al[1].

Problem 4.3

Consider the following non-linear third-order boundary value problem[1, 4]

$$y'''(x)+2e^{-3y}=4(1+x)^{-3} \tag{4.5}$$

Subject to the boundary conditions

$$y(0)=0,\ y'(0)=1,\ y'(1)=\tfrac{1}{2} \tag{4.6}$$

The exact solution is $y(x)=\ln(1+x)$. The nonlinear boundary value problem (4.5)-(4.6) is linearized by the quasilinearization technique [20] to obtain

$$y_{k+1}^{iii}-6e^{-3y_k}y_{k+1}=-6e^{-3y_k}y_k-2e^{-3y_k}+4(1+x)^{-3},\ k=0,1,... \tag{4.7}$$

$$\text{subject to}\quad y_{k+1}(0)=0,\ y_{k+1}^{i}(0)=1,\ y_{k+1}^{i}(1)=\tfrac{1}{2} \tag{4.8}$$

Using the initial approximation $y_0(x)=x-\frac{1}{4}x^2$, the numerical results for this problem at third iteration (*i.e.* k=2) at some selected points are presented in Table 3 for both n=8 and n=10, and in comparison with the results obtained in Khan and Aziz[4] and Akram et al[1].

Problem 4.4

Consider the following non-homogenous non-linear third-order boundary value problem

$$y'''(x)+\left(y''(x)\right)^2=\sin^2 x-\cos x \quad 0\le x\le 1 \tag{4.9}$$

Subject to the boundary conditions

$$y(0)=0,\ y'(0)=1,\ y'(1)=y'(\tfrac{1}{2})+\cos(1)-\cos(\tfrac{1}{2}) \tag{4.10}$$

The exact solution is $y(x)=\sin x$. Linearizing the nonlinear boundary value problem (4.9)-(4.10) by the quasilinearization technique [20], we obtain

$$y_{k+1}^{iii}+2y_k^{ii}y_{k+1}^{ii}=\left(y_k^{ii}\right)^2+\sin^2 x-\cos x,\ k=0,1,... \tag{4.11}$$

$$\text{subject to}\quad y_{k+1}(0)=0,\ y_{k+1}^{i}(0)=1,\ y_{k+1}^{i}(1)=y_{k+1}^{i}(\tfrac{1}{2})+\cos(1)-\cos(\tfrac{1}{2}) \tag{4.12}$$

With the initial approximation $y_0(x)=x+\left(\cos(1)-\cos\left(\tfrac{1}{2}\right)\right)x^2$, the numerical results for this problem at fourth iteration (*i.e.* k=3) at some selected points are presented in Table 4 for both n=8 and n=10.

Table 1: Comparison of Absolute Errors in numerical results for Example 4.1

X	Akram et al[1] (n=20)	Akram et al[1] (n=80)	Present method (n=8)		Present method (n=10)	
			Evenly-spaced nodes	Unevenly-spaced nodes	Evenly-spaced nodes	Unevenly-spaced nodes
0	0	0	2.8834E-018	5.8188E-018	2.2028E-018	1.5256E-017
0.1	8.29E-07	9.68E-10	9.9859E-008	5.8083E-009	2.5387E-010	7.0433E-012
0.2	-	-	2.3056E-007	1.4700E-008	5.5718E-010	3.9180E-013
0.3	1.63E-07	8.67E-09	3.5655E-007	3.1422E-009	8.6025E-010	3.2954E-012
0.4	4.88E-07	8.85E-09	4.8630E-007	2.3245E-009	1.1654E-009	2.1119E-011
0.5	4.62E-07	2.52E-09	6.1654E-007	1.5719E-008	1.4726E-009	1.1491E-011
0.6	-	-	7.4853E-007	3.3380E-008	1.7826E-009	2.2323E-012
0.7	8.12E-07	3.57E-09	8.8360E-007	2.6617E-008	2.0963E-009	2.1067E-011
0.8	-	-	1.0187E-006	1.4216E-008	2.4146E-009	2.4483E-011
0.9	6.60E-07	7.56E-09	1.1639E-006	2.4924E-008	2.7417E-009	1.6431E-011
1.0	0	0	1.2782E-006	3.1640E-008	3.0230E-009	2.4399E-011

Table 2: Comparison of numerical results for Example 4.2

X	Exact solution	Approximate solution Present method (n=3)	Absolute errors Present method (n=3)	Absolute errors Akram et al[1] (n=100)
0	0	0	0.00000	0.000000
0.1	1.4000E-002	1.4000E-002	0.00000	5.17E-09
0.2	5.2000E-002	5.2000E-002	0.00000	4.13E-08
0.3	1.0800E-001	1.0800E-001	0.00000	1.39E-07
0.4	1.7600E-001	1.7600E-001	0.00000	3.32E-07
0.5	2.5000E-001	2.5000E-001	0.00000	6.53E-07
0.6	3.2400E-001	3.2400E-001	0.00000	1.13E-06
0.7	3.9200E-001	3.9200E-001	0.00000	1.81E-06
0.8	4.4800E-001	4.4800E-001	0.00000	2.73E-06
0.9	4.8600E-001	4.8600E-001	0.00000	3.94E-06
1.0	5.0000E-001	5.0000E-001	0.00000	5.48E-06

Table 3: Comparison of absolute errors in numerical results for Example 4.3 at third iteration

X	Khan and Aziz[4]	Akram et al[1] (n=50)	Present method (n=8)		Present method (n=10)	
			Evenly spaced nodes	Unevenly-spaced nodes	Evenly spaced nodes	Unevenly-spaced nodes
0	0.000000	0.000000	4.3082E-017	1.6698E-017	2.0254E-018	7.7285E-017
0.1	0.0000056	3.22E-07	1.6609E-006	1.0557E-007	1.1231E-007	3.7293E-009
0.2	0.0000095	4.75E07	3.6772E-006	2.9542E-007	2.3653E-007	8.1814E-010
0.3	0.0000032	1.94E-07	5.4362E-006	1.5235E-007	3.4931E-007	1.5976E-009
0.4	0.0000175	6.28E-07	7.0699E-006	5.3985E-008	4.5258E-007	8.4219E-009
0.5	0.0000292	8.11E-07	8.5540E-006	2.4358E-007	5.4715E-007	6.0106E-009
0.6	0.0000288	4.38E-07	9.9124E-006	4.5082E-007	6.3404E-007	2.8895E-009
0.7	0.0000132	3.86E-07	1.1171E-005	4.1400E-007	7.1420E-007	7.7568E-009
0.8	0.0000051	7.77E-07	1.2312E-005	3.0403E-007	7.8849E-007	9.0095E-009
0.9	0	2.35E-07	1.3419E-005	3.7922E-007	8.5814E-007	7.0981E-009
1.0	0	0	1.4227E-005	4.3069E-007	9.1360E-007	8.7851E-009

Table 4: Comparison of absolute errors in numerical results for Example 4.4 at fourth iteration

X	Present method (n=8)		Present method (n=10)	
	Evenly-spaced nodes	Unevenly-spaced nodes	Evenly -spaced Nodes	Unevenly-spaced nodes
0	3.8937E-018	2.2549E-017	2.1090E-018	1.9638E-017
0.1	8.8540E-010	5.9524E-011	1.4332E-012	3.2474E-014
0.2	2.1237E-009	1.6797E-010	3.2857E-012	2.3759E-014
0.3	3.4167E-009	1.2344E-010	5.2944E-012	3.5638E-014
0.4	4.8387E-009	1.3939E-010	7.4797E-012	2.0095E-014
0.5	6.3711E-009	3.6888E-010	9.8521E-012	7.4940E-014
0.6	8.0310E-009	6.4213E-010	1.2429E-011	1.9429E-013
0.7	9.8410E-009	7.7079E-010	1.5233E-011	1.9218E-013
0.8	1.1793E-008	8.7081E-010	1.8293E-011	2.6756E-013
0.9	1.3972E-008	1.1518E-009	2.1654E-011	4.2333E-013
1.0	1.6161E-008	1.4655E-009	2.5138E-011	5.3380E-013

5. CONCLUSION

In this paper, we have studied a method based on Chebyshev polynomials of the first kind for solving both linear and non-linear third-order multi-point boundary value problems. Besides the advantage of high accurate results, the proposed method solves the boundary value problem without converting it to a system of first-order differential equations thereby reducing the computational cost.

REFERENCES

[1] G. Akram, M. Tehseen, S. S. Siddiqi, and H. Rehman, "Solution of a linear third order multi-point boundary value problem using RKM," *British Journal of Mathematics & Computer Science*, vol.3, no. 2, pp. 180-194, 2013.

[2] Z. Li, Y. Wang, and F. Tan, "The Solution of a class of third-order Boundary value problems by the reproducing kernel method," *Abstract and Applied Analysis,* vol. 2012, pp.1-11, 2012.

[3] F. A. Abd El-Salam, A. A. El-Sabbagh, and Z. A. Zaki, "The numerical solution of linear third order boundary value problems using non-polynomial spline technique," *Journal of American Science*, vol. 6, no. 12, pp. 303-309, 2010.

[4] A.Khan and T. Aziz, "The numerical solution of third-order boundary-value problems using quintic splines," *International Journal of Computer Mathematics*, vol. 79, no. 9, pp. 1025-1031, 2003.

[5] X. Li and B. Wu, "Reproducing kernel method for singular multi-point boundary value problems," *Mathematical Sciences*, vol. 6, no. 16, pp. 1-5, 2012.

[6] B. Wu and X. Li, "Application of reproducing kernel method to third order three point boundary value problems," *Applied Mathematics and Computation,* vol. 217, no. 7, pp. 3425–3428, 2010.

[7] A. S. Abdullah, Z. A. Majid, and N. Senu, "Solving third-order boundary value problems using fourth-order block method," *Applied Mathematical Sciences*, vol. 53, 2629-2645, 2013.

[8] F. Geng and M. Cui, "Multi-point boundary value problem for optimal bridge design," *International Journal of Computer Mathematics,* vol. 87, pp. 1051-1056, 2010.

[9] S. Timoshenko, *Theory of elastic stability*, McGraw-Hill, New York, 1961.

[10] F. Geng, "Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method," *Applied Mathematics and Computation,* vol. 215, pp. 2095-2102, 2009.

[11] Y. Lin and J. Lin, "A numerical algorithm for solving a class of linear nonlocal boundary value problems," *Applied Mathematics Letters,* vol. 23, no.9, pp. 997-1002, 2010.

[12] M. Tatari amd M. Dehghan, "The use of the Adomain decomposition for solving multi- point boundary value problems," *Physica Scripta*, vol.73, no.6, pp. 672-676, 2006.

[13] B.Y. Wu and X. Y. Li, "A new algorithm for a class of linear nonlocal boundary value problems based on the reproducing kernel method," *Applied Mathematics Letters,* vol. 24, no. 2, pp. 156-159, 2011.

[14] M.K. Kwong, "The shooting method and multiple solutions of two/multi-point BVPs of second order ODE," *Electronic Journal of Qualitative Theory of Differential Equations,* vol. 6, pp. 1-14, 2006.

[15] Y. Zou, Q. Hu, and R. Zhang, "On numerical studies of multi-point boundary value problem and its fold bifurcation," *Applied. Mathematics and Computation,* vol.185, no. 1, pp. 527-537, 2007.

[16] J. Rashidinia and M. Ghasemi, "B-spline collocation for solution of two-point boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2325-2342, 2011[17] H.N. Caglar, S.H. Caglar, and E.H. Twizell, "The numerical solution of third-order boundary value problems with fourth degree B-spline functions," *International Journal of Computer Mathematics*, vol. 71, no.3, pp. 373-381,1999.

[18] I.A. Tirmizi, E.H.Twizell, and Siraj-UI-Islam, "A numerical method for third-order boundary value problems in engineering," *International Journal of Computer Mathematics*, vol.82, no.1, pp.103-109, 2005.

[19] T. S. El-Danaf, "Quartic non-polynomial spline solutions for third order two-point boundary value problem," *World Academy of Science, Engineering and Technology*, vol. 45, pp. 453-456, 2008.

[20] R.E. Bellman and R.E. Kalaba, *Quasilinearization and nonlinear boundary value problems,* American Elsevier, New York, 1965.

[21] M. Moshiinsky, “Sobre los problem as de condiciones a la frontiera en una dimension de caracteristicas discontinuas,” *Bol. Soc. Mat. Mexicana*, vol. 7, pp. 1-25, 1950.

[22] M. Haque, M.H. Baluch, and M.F.N. Mohsen,”Solution of multipoint nonlinear boundary value problems by method of weighted residuals,” *International Journal of Computer Mathematics*, vol.18, no.3, pp.341-354, 1986.

[23] S.S. Siddiqi and M. Iftikhar, “Use of Homotopy perturbation method for solving multi-point boundary value problems,” *Research Journal of Applied Sciences, Engineering and Technology*, vol. 7, no. 4, pp. 778-785, 2014.