# A Numerical Approach for Solving First Order Integro-Differential Equations

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**Abstract**   A polynomial spline of degree $n$ is made up of polynomial segments of degree $n$ that are connected in a way that guarantees the continuity of the function and of its derivatives up to order $n-1$. This paper presents a numerical method based on cubic spline function with a free boundary condition for the solution of first order integro-differential equations. The solution procedure of this technique is simple and straightforward. Several test examples are considered to demonstrate the applicability and performance of the method. The results obtained by the proposed method are compared with the exact solutions and some existing results in literatures.

**Keywords**   Integro-differential Equation, Cubic Spline, Free Boundary Condition

## 1. Introduction

Various physical problems in sciences and engineering are modelled by integro-differential equations[16]. Integro-differential equations are usually difficult to solve analytically so there is a need to obtain an efficient approximate solution[6, 12]. Several methods have been proposed for solving these equations but most of them have their limitations such as unrealistic assumptions, linearization, low convergence and divergent results. Among these methods are Wavelet-Galerkin method[2], Chebyshev wavelets method[3], Adomain decomposition method[7], CAS wavelets method[9], Sine-Cosine wavelets[24], Homotopy perturbation method[25], Differential transform method(DTM)[6] and New Homotopy Analysis method (NHAM)[23].

Spline functions play a significant role in both the analysis and solution of a variety of problems in applied mathematics and engineering. Classes of spline functions possess many nice structural properties and excellent approximation powers. The rapid development of spline functions is due primarily to their great usefulness in applications[19]. Spline

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methods have been used to solve various forms of ordinary differential equations[4, 8, 11, 14, 26, 20 - 22], partial differential equations[5, 10] and Integral equations[13, 15, 17 - 18]. All these successful applications affirmed the validity and effectiveness of spline methods.

In this paper is presented a cubic spline method for solving first order Fredholm integro-differential equations of the form:

$$y'(x) + f(x)y(x) + \lambda \int_a^b k(x,s)y(s)\, ds = g(x) \qquad (1)$$

$$y(a) = y_0 \qquad (2)$$

where the functions $f(x)$, $g(x)$ and $k(x,s)$ are sufficiently smooth real valued functions.

## 2. Description of the Method

To develop the spline approximation method for solving the integro-differential equation (1) - (2), the interval[a,b,] is divided into n equal subintervals using the grids $x_i = a + ih$, $i = 0, 1, \cdots, n$ where $h = \dfrac{(b-a)}{n}$. By following[21], the cubic spline $S(x)$ interpolating the functions $y(x)$ at the grid points is given by the equation

$$S(x) = \frac{1}{6h}(x_i - x)^3 M_{i-1} + \frac{1}{6h}(x - x_{i-1})^3 M_i + \left(\frac{y_{i-1}}{h} - \frac{h}{6}M_{i-1}\right)(x_i - x) + \left(\frac{y_i}{h} - \frac{h}{6}M_i\right)(x - x_{i-1}) \tag{3}$$

where $M_i = S''(x_i)$ and $y_i = y(x_i)$. The unknown derivative $M_i$ are related by enforcing the continuity condition on $S'(x)$.

Differentiating (3), we obtain

$$S'(x) = \left(\frac{h}{6} - \frac{(x_i - x)^2}{2h}\right)M_{i-1} + \left(\frac{(x - x_{i-1})^2}{2h} - \frac{h}{6}\right)M_i + \frac{1}{h}(y_i - y_{i-1}) \tag{4}$$

From (4) we obtain one sided limits of the derivative as

$$S'(x_i^+) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{1}{h}(y_{i+1} - y_i), \quad i = 0, 1, \cdots, n-1. \tag{5}$$

and

$$S'(x_i^-) = \frac{h}{3}M_i + \frac{h}{6}M_{i-1} + \frac{1}{h}(y_i - y_{i-1}), i = 1, 2, \ldots, n. \tag{6}$$

The continuity condition $S'(x_i^+) = S'(x_i^-)$ gives the consistency relation

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left(y_{i-1} + 2y_i + y_{i+1}\right), \quad i = 1, 2, \cdots, n-1. \tag{7}$$

Now we collocate equation (1) at the uniform grid points $x_j = x_0 + jh$ $(j = 0, 1, \cdots, n)$ with $x_0 = a$ and $x_n = b$.

Thus (1) becomes

$$y'(x_j) + f(x_j)y(x_j) + \lambda \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} k(x_j, s)y(s)\, ds = g(x_j), j = 0, 1, \cdots, n \tag{8}$$

Using (3) to approximate the integral terms, we have

$$y'(x_j) + f(x_j)y(x_j) + \lambda \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} k(x_j, s)\left\{\left[\frac{1}{6h}(s_i - s)^3 M_{i-1} + \frac{1}{6h}(s - s_{i-1})^3 M_i\right.\right.$$
$$\left.\left. + \left(\frac{y_{i-1}}{h} - \frac{hM_{i-1}}{6}\right)(s_{i+1} - s) + \left(\frac{y_i}{h} - \frac{hM_i}{6}\right)(s - s_{i-1})\right]\right\} ds = f(x_j), j = 0, 1, \cdots, n. \tag{9}$$

Now substituting $s = s_{i-1} + ph$ and simplifying, we obtain

$$y'(x_j) + f(x_j)y(x_j) + \lambda h \sum_{i=1}^{n} \int_0^1 k(x_j, s_{i-1} + ph)\left[(1-p)y_{i-1} + py_i\right.$$
$$\left. + \frac{h^2}{6}(-2p + 3p^2 - p^3)M_{i-1} + \frac{h^2}{6}(p^3 - p)M_i\right] dp = g(x_j), \; j = 0, 1, \cdots, n. \tag{10}$$

We now split (10) into two as

$$y'(x_0) + f(x_0)y(x_0) + \lambda h \sum_{i=1}^{n} \int_0^1 k(x_0, s_{i-1} + ph)\left[(1-p)y_{i-1} + py_i\right.$$
$$\left. + \frac{h^2}{6}(-2p + 3p^2 - p^3)M_{i-1} + \frac{h^2}{6}(p^3 - p)M_i\right] dp = g(x_0) \tag{11}$$

and

$$y'(x_j) + f(x_j)y(x_j) + \lambda h \sum_{i=1}^{n} \int_0^1 k(x_j, s_{i-1} + ph)\Big[(1-p)y_{i-1}$$

$$+ py_i + \frac{h^2}{6}(-2p + 3p^2 - p^3)M_{i-1} + \frac{h^2}{6}(p^3 - p)M_i \Big] dp = g(x_j), \ j = 1, 2, \cdots, n \quad (12)$$

Putting (5) for i=0 into equation (11) we find

$$-\frac{h}{3}M_0 - \frac{h}{6}M_1 + \left(f_0 - \frac{1}{h}\right)y_0 + \frac{1}{h}y_1 + \lambda h \sum_{i=1}^{n} \int_0^1 k(x_0, s_{i-1} + ph)\Big[(1-p)y_{i-1} + py_i$$

$$+ \frac{h^2}{6}(-2p + 3p^2 - p^3)M_{i-1} + \frac{h^2}{6}(p^3 - p)M_i \Big] dp = g(x_0) \quad (13)$$

Similarly, putting (6) in (12) yields

$$\frac{h}{3}M_j + \frac{h}{6}M_{j-1} - \frac{1}{h}y_{j-1} + \left(\frac{1}{h} + f_j\right)y_j + \lambda h \sum_{i=1}^{n} \int_0^1 k(x_j, s_{i-1} + ph)\Big[(1-p)y_{i-1} + py_i$$

$$+ \frac{h^2}{6}(-2p + 3p^2 - p^3)M_{i-1} + \frac{h^2}{6}(p^3 - p)M_i \Big] dp = g(x_j), \quad j = 1, 2, \ldots, n. \quad (14)$$

Equations (13) and (14) together with (6) consist of 2n equations with 2(n+1) unknowns $y_i$, $M_i, i = 0, 1, \cdots, n$. However, to determine the values of these unknowns, two more equations are required. These equations are obtained by using the initial condition (2) and by imposing a free boundary condition

$$M_0 = 0 \quad (15)$$

# 3. Numerical Experiments

In this section the proposed method described in Section 2 is applied to some illustrative examples of first order integro-differential equations. All computations were carried out by using MATLAB 2010b.

**Example 3.1**

Consider the following linear Fredholm integro-differential equation

$$y'(x) - \int_0^1 3xsy(s) \, ds = 3e^{3x} - \frac{1}{3}\left(2e^3 + 1\right)x$$

$$y(0) = 1$$

The exact solution is $e^{3x}$

**Example 3.2**

Consider the following integro-differential equation

$$y'(x) - \int_0^1 xsy(s) \, ds = 1 - \frac{x}{3}$$

$$y(0) = 0$$

The exact solution is $x$

**Example 3.3**

Consider

$$y'(x) - \int_0^1 xy(s) \, ds = xe^x + e^x - x$$

$$y(0) = 0$$

The exact solution is $xe^x$

**Example 3.4**

Consider

$$y'(x) - y(x) - \frac{1}{(\ln 2)^2} \int_0^1 \frac{x}{s+1} y(s)\, ds = -\frac{1}{2}x + \frac{1}{x+1} - \ln(1+x)$$

$y(0) = 0$   with the exact solution $y(x) = \ln(x+1)$

**Table 1.**   Absolute Errors for Example 3.1

| $x$ | Proposed method $h = \frac{1}{10}$ | Proposed method $h = \frac{1}{20}$ |
|---|---|---|
| 0.1 | 1.4930E-02 | 2.1268E-05 |
| 0.2 | 3.0737E-04 | 8.6690E-05 |
| 0.3 | 1.5569E-02 | 1.9615E-04 |
| 0.4 | 1.2726E-03 | 3.4948E-04 |
| 0.5 | 1.6855E-02 | 5.4646E-04 |
| 0.6 | 2.8710E-03 | 7.8680E-04 |
| 0.7 | 1.8754E-02 | 1.0701E-03 |
| 0.8 | 5.0574E-03 | 1.3958E-03 |
| 0.9 | 2.1207E-02 | 1.7632E-03 |
| 1.0 | 7.7497E-03 | 2.1714E-03 |

**Table 2.**   Absolute Errors for Example 3.2

| $x$ | Proposed method $h = \frac{1}{10}$ | NHAM(N=10) [23] | Method in[1] | CAS wavelets[9] with $k = 2, m = 1$ | DTM[6] |
|---|---|---|---|---|---|
| 0.1 | 1.3878E-17 | 1.0000E-09 | 2.06509E-04 | 2.17942375E-04 | 1.66666667E-03 |
| 0.2 | 0.0000E+00 | 1.0000E-09 | 8.04069E-04 | 6.38548213E-04 | 6.09388620E-03 |
| 0.3 | 5.5511E-17 | 1.0000E-09 | 1.72624E-03 | 7.91370487E-04 | 1.32017875E-02 |
| 0.4 | 5.5511E-17 | 1.0000E-09 | 2.86044E-03 | 2.15586005E-02 | 2.29140636E-02 |
| 0.5 | 1.1102E-16 | 1.0000E-09 | 4.04527E-03 | 4.99358429E-03 | 3.51578404E-02 |
| 0.6 | 1.1102E-16 | 1.0000E-09 | 5.06663E-03 | 2.21728815E-02 | 6.69648304E-02 |
| 0.7 | 1.1102E-16 | 2.0000E-09 | 5.65279E-03 | 1.05645449E-04 | 7.12430514E-02 |
| 0.8 | 1.1102E-16 | 3.0000E-09 | 5.46844E-03 | 1.43233681E-03 | 8.63983845E-02 |
| 0.9 | 1.1102E-16 | 2.0000E-09 | 4.10753E-03 | 2.07747461E-02 | 1.08103910E-01 |
| 1.0 | 2.2204E-16 | - | - | - | 1.32023989E-01 |

**Table 3.**   Absolute Errors for Example 3.3

| $x$ | Proposed method $h = \frac{1}{10}$ | Proposed method $h = \frac{1}{20}$ | CAS wavelets[9] with $k = 2, m = 1$ | DTM[6] |
|---|---|---|---|---|
| 0.1 | 3.3341E-03 | 2.4795E-06 | 1.34917637E-03 | 1.00118319E-02 |
| 0.2 | 3.9241E-05 | 9.9525E-06 | 1.15960044E-03 | 2.78651355E-02 |
| 0.3 | 3.4132E-03 | 2.2419E-05 | 5.67152531E-03 | 5.08730892E-02 |
| 0.4 | 1.5805E-04 | 3.9878E-05 | 5.93105645E-02 | 7.55356316E-02 |
| 0.5 | 3.5717E-03 | 6.2330E-05 | 1.32330751E-02 | 9.71888592E-02 |
| 0.6 | 3.5638E-04 | 8.9774E-05 | 4.3928772E-02 | 1.09551714E-01 |
| 0.7 | 3.8098E-03 | 1.2221E-04 | 1.41201624E-02 | 1.04133232E-01 |
| 0.8 | 6.3418E-04 | 1.5964E-04 | 1.34514117E-02 | 6.94512700E-02 |
| 0.9 | 4.1273E-03 | 2.0205E-04 | 1.32045209E-02 | 1.00034260E-02 |
| 1.0 | 9.9138E-04 | 2.4946E-04 | - | 1.55147712E-01 |

**Table 4.**  Numerical Results for Example 3.4

| $x$ | Exact solution | Proposed method $h = \frac{1}{20}$ | Cheby wavelets[3] with $k = 3, m = 3, \varepsilon = 10^{-4}$ |
|---|---|---|---|
| 0.1 | 0.0953101798 | 0.0953423290 | 0.0952889907 |
| 0.2 | 0.1823215568 | 0.1823952014 | 0.8125449463 |
| 0.3 | 0.2623642645 | 0.2624897747 | 0.2624945635 |
| 0.4 | 0.3364722366 | 0.3366611089 | 0.3367692254 |
| 0.5 | 0.4054651081 | 0.4057300802 | 0.4058829389 |
| 0.6 | 0.4700036292 | 0.4703588084 | 0.4704350184 |
| 0.7 | 0.5306282511 | 0.5310892563 | 0.5312459776 |
| 0.8 | 0.5877866649 | 0.5883707849 | 0.5884913528 |
| 0.9 | 0.6418538862 | 0.6425802536 | 0.6427604341 |
| 1.0 | 0.6931471806 | 0.6940369650 | 0.6939446730 |

# 4. Conclusions

In this paper, a new numerical method has been successfully developed for solving first order linear Fredholm integro-differential equation. The method produces a system of algebraic equations which is diagonal; hence permits easy algorithm with the associated advantage of low computational cost. This method apart from being computationally efficient is also simple and reliable. The numerical results further established the superiority in terms of accuracy of the proposed method over other existing methods.

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