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SPLINE METHODS FOR A CLASS OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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ABSTRACT

In this paper, we develop numerical methods based on a non-polynomial spline function with uniform grid for solving certain class of singularly perturbed boundary value problems. The proposed methods are second-order and fourth-order accurate. Numerical examples are provided to demonstrate the efficiency of the proposed methods.

Keywords: Non-polynomial Spline; Singular perturbation; Boundary value problems.

INTRODUCTION

Consider singularly perturbed two point boundary value problems of the form:

$$-\varepsilon u'' + p(x)u = f(x), \quad 0 < p(x) \le p(1) \tag{1}$$

$$u(0) = A \qquad u(1) = B \tag{2}$$

where A and B are constants and ε is a small positive parameter such that $0 < \varepsilon << 1$ and $p(x)$ and $f(x)$ are small bounded functions.

Singularly perturbed boundary value problems arise in several branches of applied mathematics which include fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, aerodynamics, reaction diffusion process, optimal control, gas porous electrodes theory, etc. The presence of small parameter (s) ε in these problems poses some difficulties in obtaining satisfactory numerical solutions as reported in Natesan and Deb (2007). Several numerical techniques have been developed for the solution of second order singularly perturbed boundary value problems (Abrahamson and Osher, 1982; Surla and Stojanovic, 1988).

Derivation of the Methods

To solve the singularly perturbed boundary value problem (1) with (2) in the finite interval [0,1], we partition the interval using equally spaced knots $x_0 = 0,\ x_n = 1,\ x_i = ih,\ i = 1, 2,\ \cdots, n\text{-}1$ and step length $h = 1/n$, where n is an arbitrary positive integer. For each subinterval $(x_i,\ x_{i+1}),\ i = 0,1,\ \cdots, n\text{-}1$, the proposed non polynomial function has the form:

$$S(x) = a_i + b_i(x - x_i) + c_i \sin\tau(x - x_i) + d_i \cos\tau(x - x_i) \tag{3}$$

where $a_i, b_i,\ c_i,\ d_i$ are real finite non-zero constants to be determined and τ is a free parameter.

Let u_i be an approximation to $u(x_i)$, obtained by the non polynomial S passing through the points $(x_i,\ u_i)$ *and* $(x_{i+1},\ u_{i+1})$. The spline function (3) is not only required to the given differential equation and the associated boundary conditions at x_i and x_{i+1}, but it must also satisfy the continuity of first derivative at the common nodes $(x_i,\ u_i)$.

In order to derive expressions for the coefficients in (3) in terms of $u_i,\ u_{i+1},\ M_i,\ M_{i+1}$, the following relations are defined:

$$S(x_i) = u_i \quad S(x_{i+1}) = u_{i+1} \quad S''(x_i) = M_i \quad S''(x_{i+1}) = M_{i+1} \tag{4}$$

After some algebraic manipulations, we obtain from (3)

$$a_i = u_i + \frac{M_i}{\tau^2}, \quad b_i = \frac{1}{h}(u_{i+1} - u_i) + \frac{1}{\tau\theta}(M_{i+1} - M_i)$$

$$c_i = \frac{1}{\tau^2 \sin\theta}(M_i \cos\theta - M_{i+1}), \quad d_i = \frac{-M_i}{\tau^2} \tag{5}$$

where $\theta = \tau h,\ i = 0,1,\cdots n\text{-}1.$

One sided limits of the derivative of $S(x)$ are obtained as

$$S'(x_i^-) = \frac{1}{h}(u_i - u_{i-1}) + \beta h M_i + \alpha h M_{i-1},\ i = 1,2,\cdots,n. \tag{6}$$

and

$$S'(x_i^+) = \frac{1}{h}(u_{i+1} - u_i) - \alpha h M_{i+1} - \beta h M_i,\ i = 0,1,\cdots,n\text{-}1. \tag{7}$$

where $\alpha = \left(\frac{1}{\theta \sin\theta} - \frac{1}{\theta^2} \right), \qquad \beta = \left(\frac{1}{\theta^2} - \frac{\cos\theta}{\theta \sin\theta} \right)$

Using the continuity condition of the first derivative at (x_i, u_i), that is, $S'(x_i^-) = S'(x_i^+)$,

we obtain the following consistency relation:

$$u_{i-1} - 2u_i + u_{i+1} = h^2(\alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1}), \quad i = 1, 2, \cdots, n-1. \tag{8}$$

Now, setting $x = x_i$ in equation (1) and making use of (4), we have

$$M_i = \frac{p(x_i)}{\varepsilon} u_i - \frac{1}{\varepsilon} f_i \tag{9}$$

It follows also that

$$M_{i-1} = \frac{p(x_{i-1})}{\varepsilon} u_{i-1} - \frac{1}{\varepsilon} f_{i-1} \tag{10}$$

and

$$M_{i+1} = \frac{p(x_{i+1})}{\varepsilon} u_{i+1} - \frac{1}{\varepsilon} f_{i+1} \tag{11}$$

On using (9) – (11) in the consistency relation (8), we obtain

$$\begin{aligned}(\alpha h^2 p_{i-1} - \varepsilon)u_{i-1} + 2(\beta h^2 + \varepsilon)u_i + (\alpha h^2 p_{i+1} - \varepsilon)u_{i+1} \\ = \alpha h^2 f_{i-1} + 2\beta h^2 f_i + \alpha h^2 f_{i+1}, \quad i = 1, 2, \cdots, n-1\end{aligned} \tag{12}$$

Equation (12) together with the two boundary conditions (2) gives a tridiagonal set of $(n+1)$equations which can be solved for the $(n+1)$ unknowns' u_i, i=0,1,….,n.

Truncation Error

The local truncation error of (12) is given by

$$\begin{aligned}T_i = \left[\alpha h^2 p(x_{i-1}) - \varepsilon\right] u(x_{i-1}) + 2\left[\beta h^2 p(x_i) + \varepsilon\right] u(x_i) + \left[\alpha h^2 p(x_{i+1}) - \varepsilon\right] u(x_{i+1}) \\ = -\alpha h^2 f(x_{i-1}) - 2\beta h^2 f(x_i) - \alpha h^2 f(x_{i+1})\end{aligned} \tag{13}$$

Expanding in Taylor series about x_i, we obtain

$$\begin{aligned}T_i = h^2[2(\alpha+\beta) - 1]\varepsilon u_i^{ii} + \frac{h^4}{12}(12\alpha + 1)\varepsilon u_i^{iv} + \frac{h^6}{360}(30\alpha - 1)\varepsilon u_i^{vi} + O(h^8) \\ i = 1, 2, \cdots, n.\end{aligned} \tag{14}$$

Thus for arbitrary choices of α and β provided that $\alpha + \beta = \frac{1}{2}$, we obtain:

Second-order Methods

$(a)\ \alpha = 1/6,\ \beta = 1/3$

$(b)\ \alpha = 1/4,\ \beta = 1/4$

$(c)\ \alpha = 1/8,\ \beta = 3/8$

$(d)\ \alpha = 1/10,\ \beta = 2/5$

$(e)\ \alpha = 1/14,\ \beta = 3/7$

$(f)\ \alpha = 1/16,\ \beta = 7/16$

Fourth-order Method

$\alpha = 1/12,\ \beta = 5/12$

NUMERICAL EXPERIMENTS AND RESULTS

In this section, we test the new methods on two problems which have exact solutions to demonstrate the efficiency and accuracy of the methods. All computations were performed using a program written Matlab R2010a.

Problem 1

Consider the following problem

$$\varepsilon u'' + u = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x)$$
$$u(0) = u(1) = 0$$

The exact solution for this problem is

$$u(x) = \frac{e^{(x-1)/\sqrt{\varepsilon}} + e^{-x/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x)$$

Problem 1 has been solved by the proposed methods with n=16, 32, 64, 128 and $\varepsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$. The maximum absolute errors are summarized in Tables 1-7. Comparing these results with the results in Surla and Stojanovic (1988), Kadalbajoo and Bawa (1996), displayed in Tables 8 and 9 respectively, shows that methods 3.1(c) - 3.1(f) and 3.2 give better approximations. Figures 1 - 2 show the comparison of the exact and approximate solutions to this problem with different values of h and ε .

Problem 2

Consider the following problem

$$-\varepsilon u'' + \left[1 + x(1-x)\right]u = 1 + x(1-x) + \left[2\sqrt{\varepsilon} - x^2(1-x)\right]e^{(x-1)/\sqrt{\varepsilon}} + \left[2\sqrt{\varepsilon} - x(1-x)^2\right]e^{-x/\sqrt{\varepsilon}}$$

$$u(0) = u(1) = 0$$

The exact solution is given by $u(x) = 1 + (x-1)e^{-x/\sqrt{\varepsilon}} - xe^{(x-1)/\sqrt{\varepsilon}}$

The above problem has been solved by the new methods with n=16, 32, 64, 128 and

$\varepsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$. The maximum absolute errors are displayed in Tables 10 - 16. In Figures 3-4, the exact and approximate solutions to this problem are compared for some methods with different values of h and ε which confirm the accuracy of the methods.

Table-1. Observed maximum absolute errors for Problem 1 by method 3.1(a)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	7.10E-3	1.78E-3	4.45E-4	1.11E-4	2.78E-5
1/32	5.69E-3	1.42E-3	3.56E-4	8.89E-5	2.22E-5
1/64	4.07E-3	1.02E-3	2.54E-4	6.35E-5	1.59E-5
1/128	6.98E-3	1.75E-3	4 .34E-4	1.09E-4	2.71E-5

Table-2. Observed maximum absolute errors for Problem 1 by method 3.1(b)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	1.43E-2	3.56E-3	8.90E-4	2.23E-4	5.56E-5
1/32	1.14E-2	2.85E-3	7.12E-4	1.78E-4	4.45E-5
1/64	8.17E-2	2.04E-3	5.08E-4	1.27E-4	3.18E-5
1/128	1.41E-2	3.50E-3	8.68E-4	2.17E-4	5.43E-5

Table-3. Observed maximum absolute errors for Problem 1 by method 3.1(c)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	3.53E-3	8.88E-4	2.22E-4	5.56E-5	1.39E-5
1/32	2.83E-3	7.10E-4	1.78E-4	4.45E-5	1.11E-5
1/64	2.03E-3	5.08E-4	1.27E-4	3.18E-5	7.94E-6
1/128	3.54E-3	8.80E-4	2.17E-4	5.43E-5	1.36E-5

Table-4. Observed maximum absolute errors for Problem 1 by method 3.1(d)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	1.38E-3	3.54E-4	8.89E-5	2.23E-5	5.56E-6
1/32	1.12E-3	2.83E-4	7.12E-5	1.78E-5	4.45E-6
1/64	8.13E-4	2.03E-4	5.08E-5	1.27E-5	3.18E-6
1/128	1.52E-3	3.59E-4	8.73E-5	2.17E-5	5.43E-6

Table-5. Observed maximum absolute errors for Problem 1 by method 3.1(e)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	1.06E-3	2.57E-4	6.38E-5	1.59E-5	3.96E-6
1/32	8.33E-4	2.04E-4	5.09E-5	1.27E-5	3.18E-6
1/64	5.82E-4	1.45E-4	3.63E-5	9.08E-6	2.27E-6
1/128	7.61E-4	2.35E-4	6.10E-5	1.54E-5	3.87E-6

Table-6. Observed maximum absolute errors for Problem 1 by method 3.1(f)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	1.82E-3	4.48E-4	1.11E-4	2.78E-5	6.96E-6
1/32	1.44E-3	3.57E-4	8.90E-5	2.22E-5	5.56E-6
1/64	1.02E-3	2.54E-4	6.35E-5	1.59E-5	3.97E-6
1/128	1.46E-3	4.20E-4	1.07E-4	2.71E-5	6.78E-6

Table-7. Observed maximum absolute errors for Problem 1 by method 3.2

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	4.07E-5	2.53E-6	1.58E-7	9.88E-9	6.17E-10
1/32	2.01E-6	1.24E-6	7.75E-8	4.84E-9	3.02E-10
1/64	5.46E-5	3.43E-6	2.15E-7	1.34E-8	8.40E-10
1/128	1.84E-4	1.23E-5	7.69E-7	4.82E-8	3.01E-9

Table-8. Maximum absolute errors reported in Surla and Stojanovic (1988) for Problem 1

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	8.06E-3	2.02E-3	5.08E-4	1.27E-4	3.17E-5
1/32	7.11E-3	1.79E-3	4.48E-4	1.12E-4	2.80E-5
1/64	6.58E-3	1.66E-3	4.15E-4	1.04E-4	2.60E-5
1/128	6.36E-3	1.61E-3	4.03E-4	1.01E-4	2.52E-5

Table-9. Maximum absolute errors reported in Kadalbajoo and Bawa (1996) for Problem 1

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	7.09E-3	1.77E-3	4.45E-4	1.11E-4	2.78E-5
1/32	5.68E-3	1.42E-3	3.55E-4	8.89E-5	2.22E-5
1/64	4.07E-3	1.01E-3	2.54E-4	6.35E-5	1.58E-5
1/128	6.97E-3	1.75E-3	4.33E-4	1.08E-4	2.71E-5

Table-10. Observed maximum absolute errors for problem 2 by method 3.1(a)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	1.9725E-3	4.9084E-4	1.2257E-4	3.0633E-5	7.6578E-6
1/32	2.9275E-3	7.2608E-4	1.8164E-4	4.5382E-5	1.1344E-5
1/64	5.1321E-3	1.2583E-3	3.1373E-4	7.8429E-5	1.9601E-5
1/128	9.4595E-3	2.3291E-3	5.8123E-4	1.4503E-4	3.6268E-5

Table-11. Observed maximum absolute errors for problem 2 by method 3.1(b)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	3.9455E-3	9.8170E-4	2.4513E-4	6.1266E-5	1.5316E-5
1/32	5.8616E-3	1.4527E-3	3.6330E-4	9.0766E-5	2.2688E-5
1/64	1.0305E-2	2.5189E-3	6.2762E-4	1.5687E-4	3.9203E-5
1/128	1.9159E-2	4.6671E-3	1.1631E-3	2.9090E-4	7.2538E-5

Table-12. Observed maximum absolute errors for problem 2 by method 3.1(c)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	9.8916E-4	2.4560E-4	6.1295E-5	1.5317E-5	3.8290E-6
1/32	1.4713E-3	3.6349E-4	9.0848E-5	2.2693E-5	5.6720E-6
1/64	2.5886E-3	6.6030E-4	1.5695E-4	3.9220E-5	9.8010E-6
1/128	4.7962E-3	1.1696E-3	2.9092E-4	7.2532E-5	1.8135E-5

Table-13. Observed maximum absolute errors for problem 2 by method 3.1(d)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	4.0017E-4	9.8523E-5	2.4536E-5	6.1279E-6	1.5317E-6
1/32	6.0100E-4	1.4617E-4	3.6388E-5	9.0802E-6	2.2690E-6
1/64	1.0760E-3	2.5480E-4	6.2940E-5	1.5698E-5	3.9210E-6
1/128	2.0546E-3	4.7688E-4	1.1694E-4	2.9049E-5	7.2563E-6

Table-14. Observed maximum absolute errors for problem 2 by method 3.1(e)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	2.7203E-4	1.2201E-5	1.7471E-5	4.3737E-6	1.0936E-6
1/32	3.9055E-4	1.7947E-4	2.5839E-5	6.4763E-6	1.6201E-6
1/64	6.4058E-4	3.0775E-4	4.4451E-5	1.1181E-5	2.7988E-6
1/128	1.0285E-3	3.1209E-4	8.1720E-5	2.0636E-5	5.1760E-6

Table-15. Observed maximum absolute errors for problem 2 by method 3.1(f)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	4.8189E-4	1.2201E-4	3.0598E-5	7.6554E-6	1.9143E-6
1/32	6.9973E-4	1.7947E-4	4.5282E-5	1.1338E-5	2.8355E-6
1/64	1.1744E-4	3.0775E-4	7.8000E-5	1.9580E-5	4.8986E-6
1/128	1.9812E-3	5.5805E-4	1.4376E-4	3.6160E-5	9.0609E-6

Table-16. Observed maximum absolute errors for problem 2 by method 3.2

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$
1/16	7.9343E-6	4.7902E-7	3.1084E-8	1.9432E-8	1.2146E-10
1/32	2.2195E-5	1.3932E-6	8.7462E-8	5.4679E-9	3.4177E-10
1/64	7.3085E-5	4.6055E-6	2.8879E-7	1.8089E-8	1.1307E-9
1/128	2.4975E-4	1.6298E-5	1.0297E-6	6.4479E-8	4.0339E-9

Fig. 1: Comparison of solutions for Problem 1, Method 3.1(a), $\varepsilon=\frac{1}{32}, h=\frac{1}{32}$

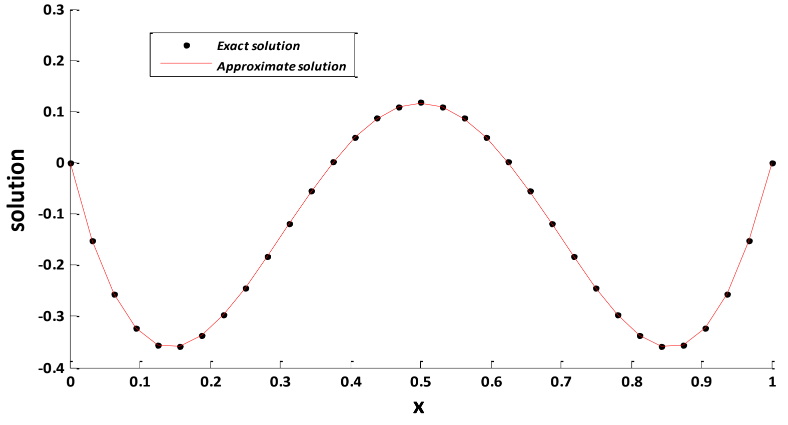


Fig. 2: Comparison of solutions for Problem 1, Method 3.2, $\varepsilon=\frac{1}{64}, h=\frac{1}{64}$

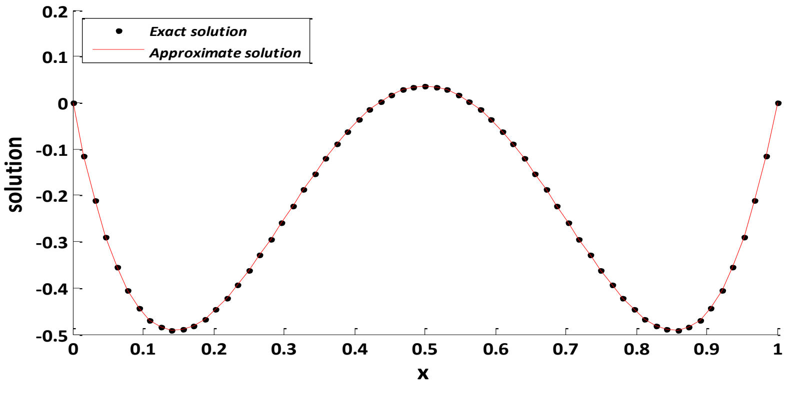


Fig. 3: Comparison of solutions for Problem 2, Method 3.1(f), $\varepsilon=\frac{1}{16}, h=\frac{1}{64}$

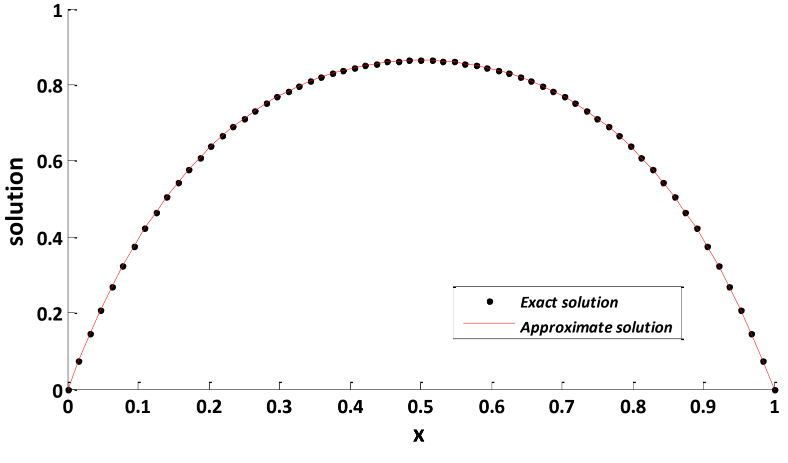
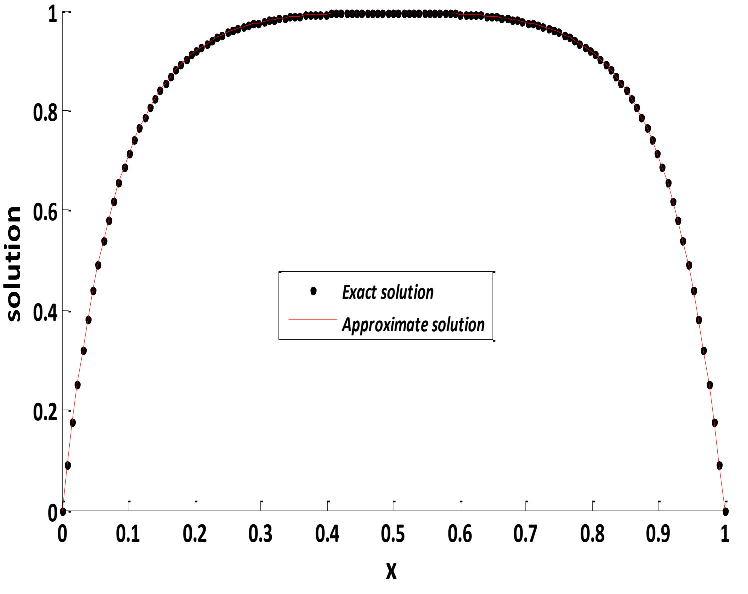


Fig. 4: Comparison of solutions for Problem 2, Method 3.2, $\varepsilon = \frac{1}{128}$, $h = \frac{1}{128}$



CONCLUSION

We have developed a class of methods based on a non-polynomial spline function for solving linear singularly perturbed boundary value problems. The methods are second-order and fourth-order accurate. Numerical results show better performance of the new methods, except methods 3.1(a) and 3.1(b), over the existing methods reported in Surla and Stojanovic (1988), Kadalbajoo and Bawa (1996). The methods are computationally efficient and the scheme can easily be implemented on a computer without wastage of computer time.

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