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On the Application of Sumudu Transform Series Decomposition Method and Oscillation Equations

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, a new method called Sumudu Transform Series Decomposition Method (STSDM) is applied to three different models of Oscillatory problems (Van der Pol, Duffing and Nonlinear Oscillatory equations). The method was developed by Combining the Sumudu Transform, Series Expansion Schemes and Adomian Polynomials. The Sumudu Transform was used to avoid integration of some difficult functions or rigour of reducing order of differential equations to system of differential equations, the Series Expansion was employed to increase the rate of convergence of the solution while Adomian Polynomials were used to decompose the nonlinear terms of the differential equations. The results obtained in all the problems considered showed that the new method was very effective, accurate and reliable.

Keywords: Adomian polynomial; duffing equation; oscillatory equation; series expansion; sumudu transform method; van der Pol equation.

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1 Introduction

Here in this section, we present brief introduction of the combined methods

1.1 Sumudu transform

Sumudu Transform is an integral-based transform named by Watugula [1]. Since the formulation of the method, many researchers have worked tirelessly using the transform to obtained results of many physical problems and thereby reported that the transform was a powerful tool for obtaining a convergence solution of many differential equations [2-6].

Sumudu Transform is written as

$$F(u) = S[f(x)] = \int_0^\infty \frac{1}{u} f(x) e^{-x/u} dx \tag{1}$$

for any f(x).

By the conversion rule

$$F(u) = \sum_{n=0}^{\infty} n! a_n u^n \tag{2}$$

for function f(x) which can be expressed as a polynomial or as a convergent infinite series for $x \geq 0$. Likewise the derivative property of Sumudu transform is given as:

Let $f(x)$ be a continuous valued-function. Then, the Sumudu Transform of m^{th} derivative ($f^{(m)}(x)$) of $f(x)$ for $m \geq 1$ is given as:

$$S\left[f^{(m)}(x)\right] = \frac{1}{z^m}\left\{S[f(x)] - f(0) - zf'(0) - \ldots - z^{(m-1)} f^{(m-1)}(0)\right\} \tag{3}$$

It can be applied to the solution of ordinary convergent equations and control engineering problems.

Among others, the Sumudu transform was shown to have units preserving properties, and hence may be used to solve problems without resorting to the frequency domain. This is one of the strength for this new transform, especially with respect to applications in problems with physical dimensions. In fact, the Sumudu transform which is itself linear preserves linear functions, and hence in particular, does not change units [1,7].

1.2 Series expansion

In the 14th century, the earliest examples of the use of Taylor series and closely related methods were given by Madhava of Sangamagrama termed [8-9]. Though no record of his work survived later the writings of Indian mathematicians suggested that he found a number of special cases of the Taylor series, including those for the trigonometric functions of sine, cosine, tangent and arctangent. The Kerala school of astronomy and mathematics further expanded his works with various series expansions and rational approximations till the 16th century.

1.3 Adomian decomposition

In 1980s, George Adomian introduced a new method to solve nonlinear differential equations [10-12]. This method has since been termed the Adomian Decomposition Method (ADM) and has been the subject of many investigations such as [13-21]. The ADM involves separating the equation under investigation into linear and nonlinear portions. The linear operator representing the linear portion of the equation is inverted and the inverse operator is then applied to the equation under any considerable given conditions.

The nonlinear portion is decomposed into a series called Adomian polynomials. This method generated a solution in form of a series whose terms are determined by a recursive relationship using these Adomian polynomials.

In this study, STSDM is applied to solve the oscillation equations considered by [22-24] and the results obtained are in excellent agreement with the existing results.

2 Mathematical Formulation of Sumudu Transform Series Decomposition Method

Derivation of the Sumudu Transform Series Decomposition Method (STSDM)

Given a general nonlinear non-homogeneous differential equation

$$Ly(x) + Ry(x) + Ny(x) = g(x) \tag{4}$$

where L is the highest order linear differential operator, R is the linear differential operator of order less than L, N is the nonlinear differential operator, U is the dependent variable, x is an independent variable and g(x) is the source term which is assumed to have series expansion.

Application of the Sumudu Transform on equation (4) resulted into

$$S[Ly(x)] + S[Ry(x)] + S[Ny(x)] = S[g(x)] \tag{5}$$

Using the differentiation property of the Sumudu transform (3) in (5) to have

$$\frac{S[y(x)]}{u^m} - \sum_{k=o}^{m-1} \frac{y(x)^{(k)}(0)}{u^{(m-k)}} + S[Ry(x)] + S[Ny(x)] = S[g(x)] \tag{6}$$

where

$$\sum_{k=o}^{m-1} \frac{y(x)^{(k)}(0)}{u^{(m-k)}} = \sum_{k=o}^{m-1} \frac{y(0)^{(k)}}{u^{(m-k)}}$$

Further simplification of (6) gave

$$S[y(x)] - u^m \sum_{k=o}^{m-1} \frac{y(x)^{(k)}(0)}{u^{(m-k)}} + u^m [S[Ry(x)] + S[Ny(x)] - S[g(x)]] = 0 \tag{7}$$

where S denotes the sumudu transform,

Application of Sumudu inverse Transform on (7) yielded

$$y(x) = G(x) - S^{-1}\left[u^m\left[[S[Ry(x)] + S[Ny(x)]]\right]\right] \tag{8}$$

and,

$$G(x) = S^{-1}\left[u^m\left[\sum_{k=o}^{m-1}\frac{y(x)^{(k)}(0)}{u^{(m-k)}} + S[g(x)]\right]\right] \tag{9}$$

Where G(x) represents the term arising from the source term and the prescribed initial conditions.

The representation of the solution (8) as an infinite series is given below:

$$y(x) = \sum_{n=o}^{\infty} y_n(x) \tag{10}$$

The nonlinear term is being decomposed as:

$$Ny(x) = \sum_{i=0}^{\infty} A_n\left(y_0(x), y_1(x), \ldots y_n(x)\right) \tag{11}$$

Where A_n are the Adomian polynomials of functions y_0, y_1, $y_2 \ldots y_n$ and can be calculated by formula given in [25] as:

$$A_n(y_0(x), y_1(x) \ldots y_n(x) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\sum_{i=0}^{\infty}\lambda^i y_i(x)\right]_{\lambda=0} \qquad n = 0, 1, 2, \ldots \tag{12}$$

Substituting (10) and (11) into (8) yielded

$$\sum_{n=o}^{\infty} y_{n+1}(x) = G(x) - S^{-1}\left[Su^m\left[[[R\sum_{n=o}^{\infty} y_n(x)] + [\sum_{n=o}^{\infty} A_n]]\right]\right] \tag{13}$$

Simplification of equation (13) as many times as possible resulted into series solution and generally recursive relation given by:

$$y_0(x) = G(x) = S^{-1}\left[u^m\left[\sum_{k=o}^{m-1}\frac{y(x)^{(k)}(0)}{u^{(m-k)}} + S[g(x)]\right]\right] \qquad n \geq 0 \tag{14}$$

$$y_{n+1}(x) = -S^{-1}\left[Su^m\left[[[Ry_n(x)] + [A_n]]\right]\right] \tag{15}$$

when the Sumudu Transform and the Sumudu inverse Transform are applied on (15) respectively, the iteration y_0, y_1, $y_2 \ldots y_n$ were obtained, which in turn gave the general solution as

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \ldots \tag{16}$$

3 Numerical Application

In this section three different types of oscillation problems are solved by the new method (STSDM).

3.1 Van Der Pol's equation

Consider the Van Der Pol's equation considered by [15] given as:

$$\frac{d^2x(t)}{dt^2}+\frac{dx(t)}{dt}+x(t)+x^2(t)\frac{dx(t)}{dt}=2\cos t-\cos^3 t \tag{17}$$
$$x(0)=0, \qquad x'(0)=1$$

The truncated Taylor series expansion of $f(t)$ is given as

$$f(t)=1+\frac{t^2}{2}-\frac{19t^4}{24}+\frac{181t^6}{720}+\ldots \tag{18}$$

Substituting (18) into (17) gave

$$\frac{d^2x(t)}{dt^2}+\frac{dx(t)}{dt}+x(t)+x^2(t)\frac{dx(t)}{dt}=1+\frac{t^2}{2}-\frac{19t^4}{24}+\frac{181t^6}{720}+\ldots \tag{19}$$

Finding the STSDM of (19) resulted into

$$\frac{1}{u^2}\left[S\left[x(t)\right]-x(0)-ux'(0)\right]=1+u^2-19u^4+181u^6+\ldots-S\left[\left(\frac{dx(t)}{dt}+x(t)+x^2(t)\frac{dx(t)}{dt}\right)\right] \tag{20}$$

Substituting the initial condition in (17) into (20) and simplifying gave

$$S\left[x(t)\right]=u+u^2+u^4-19u^6+181u^8+\ldots-u^2S\left[\left(\frac{dx(t)}{dt}+x(t)+x^2(t)\frac{dx(t)}{dt}\right)\right] \tag{21}$$

Finding the Sumudu inverse of (21) resulted into

$$x_0(t)=t+\frac{1}{2}t^2+\frac{1}{24}t^4-\frac{19}{720}t^6+\ldots. \tag{22}$$

$$\left.\begin{aligned} x_1(t)&=-S^{-1}\left[u^2S\left[\left(\frac{dx_0(t)}{dt}+x_0(t)+A_0\frac{dx_0(t)}{dt}\right)\right]\right]\\ x_2(t)&=-S^{-1}\left[u^2S\left[\left(\frac{dx_1(t)}{dt}+x_1(t)+A_1\frac{dx_1(t)}{dt}\right)\right]\right]\end{aligned}\right\} \tag{23}$$

and,

$$x_{n+1}(t)=-S^{-1}\left[u^2S\left[\left(\frac{dx_n(t)}{dt}+x_n(t)+A_n\frac{dx_n}{dt}\right)\right]\right] \tag{24}$$
$$n\geq 0$$

$A_n = x_n^2(t)\dfrac{dx_n(t)}{dt}$ is the nonlinear part that can be decomposed by (12) as follow:

$$\left.\begin{aligned}
&A_0 = x_0^2(t)\frac{dx_0(t)}{dt}\\
&A_1 = \left(\left(2x_0(t)x_1(t)\right)\frac{dx_1(t)}{dt}\right)\\
&.\\
&.\\
&.
\end{aligned}\right\} \tag{25}$$

Using the Adomian polynomial (24) in (25) and iterating gave the approximate solution

$$\left.\begin{aligned}
&x_1(t) = -\frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{1}{8}t^4 - \frac{13}{120}t^5 - \frac{31}{720}t^6 - \frac{191}{40320}t^8 + \frac{23}{12960}t^9 + \frac{71}{28800}t^{10} + \\
&\frac{1}{1920}t^{11} + \frac{19}{103680}t^{12} + \frac{223}{2695680}t^{13} - \frac{361}{7257600}t^{14} - \frac{209}{10368000}t^{15} + \ldots.\\
&x_2(t) = \frac{1}{6}t^3 + \frac{1}{8}t^4 + \frac{1}{24}t^5 - \frac{1}{90}t^6 - \frac{3}{70}t^7 - \frac{43}{1120}t^8 - \frac{1031}{36288}t^9 - \frac{967}{48384}t^{10} - \\
&\frac{210103}{19958400}t^{11} - \frac{191833}{39916800}t^{12} - \frac{10903}{5443200}t^{13} - \frac{25489}{62899200}t^{14} + \frac{603259}{6604416000}t^{15}\\
&+ \frac{32899}{261273600}t^{16} + \ldots.
\end{aligned}\right\} \tag{26}$$

The general solution x(t) is given as;

$$\left.\begin{aligned}
&x(t) = x_0(t) + x_1(t) + x_2(t)\\
&x(t) = t - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{15}t^5 - \frac{29}{360}t^6 - \frac{257}{5040}t^7 - \frac{1739}{40320}t^8 - \frac{179}{6720}t^9\\
&- \frac{21193}{1209600}t^{10} - \frac{49927}{4989600}t^{11} - \frac{1139}{246400}t^{12} - \frac{543541}{283046400}t^{13} - \frac{85853}{188697600}t^{14}\\
&- \frac{235063}{3302208000}t^{15} - \frac{32899}{261273600}t^{16} + \ldots
\end{aligned}\right\} \tag{27}$$

3.2 Duffing equation

Let's consider the Duffing equation examined by [16]

$$\begin{gathered}
x''(t) + 2x'(t) + x(t) + 8x^3(t) = e^{-3t}\\
x(0) = \frac{1}{2}, \qquad x'(0) = -\frac{1}{2}
\end{gathered} \tag{28}$$

Following the same procedure, the general solution of (28) is obtain as

$$\left.\begin{aligned}
&x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t)\\
&x(t) = \frac{1}{2} - \frac{1}{2}t + \frac{1}{4}t^2 - \frac{1}{12}t^3 + \frac{1}{48}t^4 - \frac{1}{240}t^5 + \frac{1}{1440}t^6 + \ldots.
\end{aligned}\right\} \tag{29}$$

3.3 Nonlinear oscillatory system equation

The nonlinear oscillatory system equation given by [17] is also considered here

$$x''(t) + x(t) + 0.1x^3(t) = 0 \qquad (30)$$
$$x(0) = 1, \qquad x'(0) = 0$$

The general solution of (30) is obtained by the same procedure as

$$\left.\begin{aligned} &x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t) + x_4(t) \\ &x(t) = 1 - 0.55000000t^2 + 0.0595833333t^4 - 0.005606944443t^6 \\ &+0.0007783754961t^8 + \ldots \end{aligned}\right\} \qquad (31)$$

Since one of the major characteristics of an oscillatory problem is its ability to exhibit periodicity therefore (31) cannot exhibit periodicity on its own, and to make it exhibit periodicity three steps are taken which are:

1. Find the Laplace transform of (31)
2. Find the diagonal Pade approximation of solution of step one
3. Obtain the Laplace inverse transform of the result in step two

Following all the itemized steps above, (31) is obtain as:

$$y(t) = 0.01341156746 + 0.002118710698\cos(3.202441315t) + 0.9844697222\cos(1.045887839t) \quad (32)$$

4 Numerical Results

Table 1 displays the comparison of results obtained for Van der Pol's equation by STSDM with the exact, New Algorithm for the Decomposition Solution (NADS) and the Adomian Decomposition Method (ADM), Table 2 shows the comparison of results obtained for Duffing equation by STSDM with the exact and the Differential Transform Method (DTM) and Table 3 displays the comparison of results obtained for Nonlinear Oscillatory system equation by STSDM with the exact and Differential Transform Method (DTM) while Figs.1 is the graphical representation of the solutions of nonlinear oscillatory system equation

Table 1. Comparison between STSDM with Exact (E), the New Algorithm for the Decomposition Solution (NADS) and the Adomian Decomposition Method (ADM) for Eq. (17)

T	Exact	STSDM	NADS	ADM	E-STSDM	E-NADS	E-ADM
0.0	0	0	0	0	0	0	0
0.2	0.1986693	0.1987061	0.1987475	0.1987510	0.0000368	0.0000782	0.0000817
0.4	0.3894183	0.3892662	0.3909898	0.3912929	0.0001521	0.0015715	0.0018746
0.6	0.5646424	0.5578822	0.5750719	0.5797338	0.0067602	0.0104295	0.0150914

Table 2. Comparison between the STSDM with the Exact and Numerical Solution of Duffing Equation by the Differential Transform Method for Eq. (28)

t	Exact	STSDM	DTM	E-STSDM	E-DTM
0.1	0.4524187090	0.4524187090	0.4524187092	$1.8*10^{-11}$	$2*10^{-10}$
0.2	0.4093653764	0.4093653764	0.4093653767	$6.1*10^{-11}$	$3*10^{-10}$
0.3	0.3704091103	0.3704091103	0.3704091102	$6.0*10^{-11}$	$1*10^{-10}$
0.4	0.3351600229	0.3351600229	0.3351600228	$1.9*10^{-11}$	$1*10^{-10}$
0.5	0.3032653301	0.3032653298	0.3032653298	$5.9*10^{-11}$	$3*10^{-10}$
0.6	0.2744058181	0.2744058181	0.2744058180	$9.0*10^{-11}$	$1*10^{-10}$
0.7	0.2482926519	0.2482926521	0.2482926520	$2.3*10^{-10}$	$1*10^{-10}$
0.8	0.2246644819	0.2246644830	0.2246644819	$1.1*10^{-09}$	0
0.9	0.2032848295	0.2032848334	0.2032848297	$3.7*10^{-09}$	$2*10^{-07}$

1.0	0.1839397196	0.1839397321	0.1839397202	$1.2*10^{-08}$	$6*10^{-07}$

Table 3. Comparison between the STSDM with exact and differential transform method for Eq. (30)

T	Exact	STSDM	DTM	E-STSDM	E-DTM
0	0.9989671200	1.000000000	1.00000012	0.00103288036	0.0010330000
0.1	0.9887208962	0.9945059532	0.9945060906	0.00578505696	0.005785194359
0.2	0.9676087087	0.9780949769	0.9780951673	0.01048626816	0.01048645863
0.3	0.9358625784	0.9509785885	0.9509788635	0.01511601006	0.01511628513
0.4	0.8938313928	0.9135028684	0.9135032383	0.01967147556	0.01967184549
0.5	0.8419770716	0.8661393040	0.8661396913	0.02416223236	0.02416261970
0.6	0.7808694894	0.8094729449	0.8094730032	0.02860345546	0.02860351378
0.7	0.7111802136	0.7441887223	0.7441874427	0.03300850866	0.03300722911
0.8	0.6336751239	0.6710568705	0.6710518542	0.03738174656	0.03737673034
0.9	0.5492059951	0.5909183755	0.5909046435	0.04171238036	0.04169864839
1.0	0.4587011362	0.5046712732	0.5046395018	0.04597013696	0.04593836558

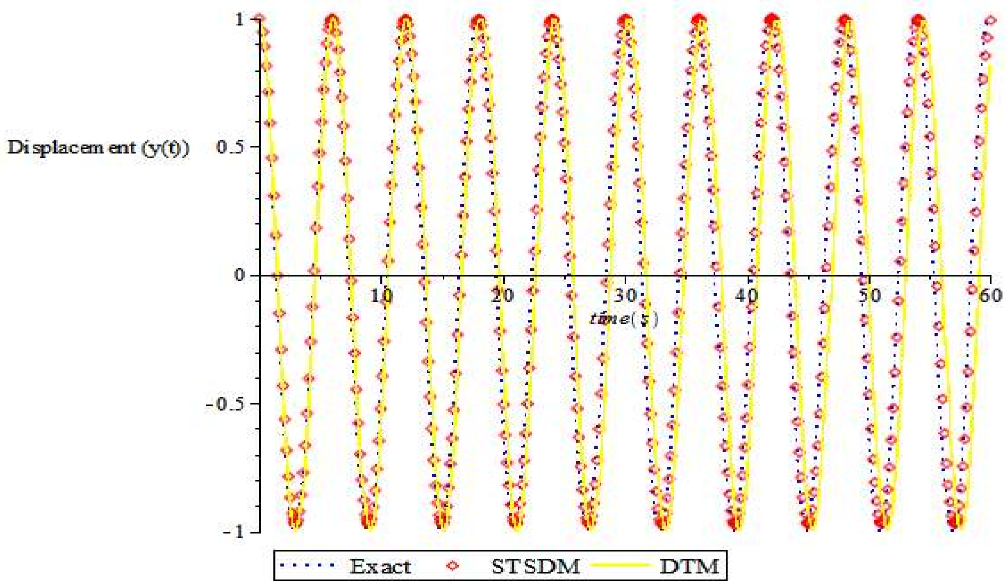

Fig. 1. Graph of displacement against time for the solution Eq. (30)

5 Conclusion

In this work we presented an alternative method of solving Van Der Pol's, Duffing and Nonlinear Oscillatory system equations called Sumudu Transform Series Decomposition Method. The method offers significant advantages in terms of its easiness, straightforward applicability, its computational effectiveness and its accuracy. The comparison of the results obtained by the Sumudu Transform Series Decomposition Method, with the exact, New Algorithm for the Decomposition Solution (NADS), Adomian Decomposition Method (ADM), and Differential Transform Method (DTM) showed that STSDM gives a better approximation and at the same time it is capable of speeding up the rate of convergence of the solution.

Competing Interests

Authors have declared that no competing interests exist.

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